Pre-course in Mathematics

FaRaFIN Vorkurs-Team

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Contents

1 Cheat Sheet

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This „cheat sheet" contains basic definitions and spellings that you will need during your studies and in the pre-course. We will usually assume that you have existing knowledge on the content of the course.

1.1 Number sets

Symbol	Description	Examples
$\mathbb{N}$	Natural numbers: positive integers. The 0 is usually included only if the notation $\mathbb{N}_0$ is used	1; 2; 3; 454647; 8892349823
$\mathbb{Z}$	Integers: All positive and negative integers	-2; -1; 0; 1; 2; 42; -645631; 3469079
$\mathbb{Q}$	Rational numbers: numbers that can be expressed as a fraction of two integers	$\frac{1}{2}$; $\frac{1}{3}$; $\frac{4}{3}$; $-\frac{6}{23}$; $0.2 (= \frac{1}{5})$
$\mathbb{R}$	Real numbers	$1.27; \sqrt{2}; \pi$
$\mathbb{C}$	Complex numbers (see chapter 10)	$2 + 3i; i; -6 - 42i$

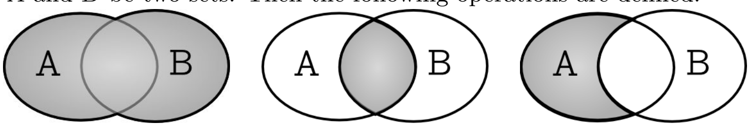
Every number set contains all number sets above it: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

1.2 Sets

Sets can be represented in different ways. These are the two most important ones:

- Explicit listing: $M = \{a, b, c, d\}$ contains the elements a, b, c, and d.
- Specification of a condition to be fulfilled: $M = \{x \in \mathbb{N} \mid 0 < x < 42\}$ contains all natural numbers between 0 and 42 (excluding these two numbers).

Let A and B be two sets. Then the following operations are defined:



union $A \cup B$ | intersection $A \cap B$ | difference $A \setminus B$

- *a is an element of* A: $a \in A$.
- The *empty set* ($\emptyset$) is the set that has no elements.
- Two sets are equal ($A = B$) if they both contain the same elements.

- Two sets are called *disjoint* if they have no common elements: $A \cap B = \emptyset$.
- A set A can be completely contained in another set B: $A \subseteq B$ (read: *A is a subset of B*). If $A \neq B$ holds, A is a *proper subset* of B ($A \subset B$).
- Analogously, A is a *(proper) superset* of B if the following holds: $A \supseteq B$ ($A \supset B$).
- The *compliment of a set* $\overline{A}$ of the set A contains all elements that are not contained in A. If A is a subset of a carrier set X (or underlying set X), then: $\overline{A} = X \setminus A$

small	large	Name	Common use
α		Alpha	angle
β		Beta	angle
γ	Γ	Gamma	angle
δ	Δ	Delta	δ: angle; Δ: difference
ε		Epsilon	very small positive number
η		Eta	
θ		Theta	θ: angle
λ		Lambda	multiplicative factor
μ		My	
ξ		Xi	
π	Π	Pi	$\pi = 3.14\ldots$; Π: product
ρ		Rho	
σ	Σ	Sigma	Σ: sum
τ		Tau	
φ	Φ	Phi	φ: angle (in polar coordinates)
χ		Chi	
ψ	Ψ	Psi	
ω	Ω	Omega	

Table 1.1: Selection of important Greek letters

1.3 Intervals

An interval is a contiguous range of numbers that is defined by its two endpoints. A distinction is made between closed and open intervals. A *closed interval* $[a, b]$ contains a and b (inclusive), while an *open interval* (a, b) does not contain a and b (exclusive).

It is possible to combine both types. This results in a *half-open interval*: $[a, b)$ contains a but not b, while $(a, b]$ contains b but not a.

Symbol	Meaning
$\exists x$	there exists (at least) one x
$\nexists x$	there does not exist any x
$\forall x$	for all x holds ...
$\pm$	plus/minus, e.g. $x_{1,2} = \pm 1 \rightarrow x_1 = -1, x_2 = +1$
$\sum\limits_{i=1}^{n} a_i$	$a_1 + a_2 + \cdots + a_n$
$\prod\limits_{i=1}^{n} a_i$	$a_1 \cdot a_2 \cdot \ \cdots \ \cdot a_n$
∞	infinite
$\wedge$	logical and
$\vee$	logical or
$\neg$	logical negation
$:=$	is defined as
$\equiv$	is equivalent to
$<$	less than (often also: „strictly less than“)
$\leq$	less than or equal to
$>$	greater than (often also: „strictly greater than“)
$\geq$	greater than or equal to
$=$	equal to
$\neq$	not equal to
$\mid$	divides
$\nmid$	does not divide

Table 1.2: Important special characters

1.4 Abbreviations and vocabulary

iff (gdw. in German) Short for „if and only if“. The symbol $\Leftrightarrow$ is also used.

qed At the end of a proof. Latin „quod erat demonstrandum“ („that which was to be shown / proved“). Meaning: Hurray, we've finally got through the proof. In print the symbol $\square$ is used.

commutative „interchangeable“. An operation (e.g. $+, \cdot$) is commutative if the two operands can be interchanged without changing the result. Expressed as a formula, this means: $a \circ b = b \circ a$, where $\circ$ stands for the operation.

distributive Factoring out is allowed: $a \cdot (b + c) = a \cdot b + a \cdot c$

associative The order in which the operation is performed is arbitrary: $a + b + c = (a + b) + c = a + (b + c)$.

there exists There is *at least one* element that fulfills the statement.

there exists exactly one There is only one element that fulfills the statement.

necessary condition This condition is always fulfilled if a statement is true. However, there are also cases in which the condition is fulfilled even though the statement is not true.

sufficient condition If this condition is met, the statement is true in any case. However, there are cases in which the statement is true but the condition is not met.

necessary and sufficient condition Whenever this condition is met, the statement is also true (and vice versa).

2 Basic Mathematics

2.1 Fractions

Author: Katja Matthes

2.1.1 Definition

A fraction is the representation of a rational number as a quotient.

Fraction: $\frac{Z}{N}$ with $Z \in \mathbb{Z}$ and $N \in \mathbb{Z} \setminus \{0\}$

Z ... numerator N ... denominator

Two fractions, $\frac{a}{b}$ and $\frac{c}{d}$, are said to have the same name if they have the same denominator: $b = d$.

2.1.2 Shortening and extending

A fraction is shortened by dividing both the numerator and the denominator by the same number.

$$\frac{a \cdot c}{b \cdot c} \overset{:c}{=} \frac{a}{b}$$

A fraction is expanded by multiplying both the numerator and the denominator by the same factor.

$$\frac{a}{b} \overset{\cdot c}{=} \frac{a \cdot c}{b \cdot c}$$

2.1.3 Special rules of calculation

Adding fractions with the same denominator

Two fractions with the same denominator are added by adding their numerators and taking the denominator.

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

Subtraction of fractions with the same denominator

Two fractions with the same denominator are subtracted by subtracting their numerators and keeping the denominator.

$$\frac{a}{b} - \frac{c}{b} = \frac{a - c}{b}$$

Multiplication by a factor

A fraction is multiplied by a factor n by multiplying the numerator by this factor and keeping the denominator.

$$\frac{a}{b} \cdot n = \frac{a \cdot n}{b}$$

Division by a number

A fraction is divided by a number $n \neq 0$ by multiplying the denominator by this number and keeping the numerator.

$$\frac{a}{b} : n = \frac{a}{b \cdot n}$$

2.1.4 General rules of arithmetic

Addition

Two fractions are added by first making them equal and then adding the numerators.

$$\frac{a}{b} + \frac{c}{d} = \frac{a \cdot d}{b \cdot d} + \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d + b \cdot c}{b \cdot d}$$

Subtraction

Two fractions are subtracted by first making them equal and then subtracting the numerators.

$$\frac{a}{b} - \frac{c}{d} = \frac{a \cdot d}{b \cdot d} - \frac{b \cdot c}{b \cdot d} = \frac{a \cdot d - b \cdot c}{b \cdot d}$$

Multiplication

Two fractions are multiplied by multiplying the denominators and numerators.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Division

A fraction is divided by another by multiplying it by its reciprocal.

$$\frac{a}{b} : \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

2.1.5 Exercises

Exercise 1

Calculate and simplify as far as possible.

1. $\frac{\frac{8}{9}}{\frac{16}{27}}$
2. $\frac{2\frac{1}{3}}{1\frac{1}{6}}$
3. $\frac{5\frac{1}{2}}{\frac{11}{12}}$
4. $\frac{\frac{99}{100}}{\frac{9}{10}}$

Exercise 2

Calculate and simplify as far as possible.

1. $\frac{5}{6} \cdot \frac{2}{3} - \frac{2}{9} + \frac{3}{4} \cdot 1\frac{7}{9}$
2. $3\frac{5}{12} - 2\frac{5}{6} + 1\frac{1}{3} : \frac{4}{9} - 2\frac{1}{6} \cdot \frac{1}{2}$

Exercise 3

Calculate and simplify as far as possible.

1. $\left(\frac{2}{3} - \frac{1}{6}\right) \cdot \left(\frac{9}{11} - \frac{3}{7}\right)$
2. $\left(\frac{1}{8} + \frac{7}{12}\right) : \left(5 - \frac{3}{4}\right)$
3. $\frac{4}{7} \cdot \left(\left(1\frac{1}{2} - \frac{5}{9}\right) : 4\frac{1}{4}\right)$
4. $\frac{4}{5} : \left[\left(\frac{5}{8} - \frac{1}{3}\right) \cdot 12\right]$
5. $\frac{3}{4} \cdot \left(2\frac{1}{2} : 1\frac{1}{4}\right)$

Exercise 4

Calculate and simplify as far as possible.

1. $\frac{\left(\frac{3}{8} \cdot \frac{2}{7}\right)}{\frac{5}{14}}$
2. $\frac{1\frac{3}{4} + \frac{5}{6}}{\frac{1}{4}}$
3. $\frac{\frac{8}{9}}{3\frac{1}{3} + \frac{1}{6}}$
4. $\frac{\left(\frac{3}{5} - \frac{5}{10}\right) : \frac{2}{5}}{\frac{1}{4} + \frac{1}{2}}$

2.2 Powers and Exponents

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2.2.1 Definition

Powers are an abbreviated way of writing repeated multiplication by a factor.

$$\underbrace{a \cdot a \cdot a \cdots a}_{n \text{ factors}} = a^n$$

a^n ... power $\qquad$ a ... base $\qquad$ n ... exponent

2.2.2 Special exponents

Let $a \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}_0$, then:

$$a^0 = 1$$

$$a^1 = a$$

$$a^{-n} = \frac{1}{a^n}$$

2.2.3 Power laws

The following power laws apply to all m, $n \in \mathbb{Z}$ and a, $b \in \mathbb{R} \setminus \{0\}$.

1. Powers with the same base are multiplied by retaining the base and adding the exponents.

   $$a^m \cdot a^n = a^{m+n}$$

2. powers with the same base are divided by retaining the base and subtracting the exponents.

   $$\frac{a^m}{a^n} = a^{m-n}$$

3. powers with the same exponent are multiplied by multiplying the bases and retaining the exponent.

   $$a^n \cdot b^n = (a \cdot b)^n$$

4. Powers with the same exponent are divided by dividing the bases and keeping the exponent.

   $$\frac{a^n}{b^n} = (\frac{a}{b})^n$$

5. Powers are raised to a power by keeping the base and multiplying the exponents.

   $$(a^m)^n = a^{m \cdot n} = a^{n \cdot m} = (a^n)^m$$

2.2.4 Roots

Let $m, n \in \mathbb{N}$ and $a \in \mathbb{R}$ with $a > 0$. Then

$$sqrt[n]a^m = a^{\frac{m}{n}}$$

Thus, the power laws also apply to roots. a is called the radicand and n the root exponent.

2.2.5 Root laws

For $m, n \in \mathbb{N}$ with $m, n > 1$ and non-negative real radicands a and b the following applies:

1. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$

2. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$

3. $\sqrt[n]{\sqrt[m]{a}} = \sqrt[mn]{a} = \sqrt[m]{\sqrt[n]{a}}$

4. $\sqrt[m]{a} \cdot \sqrt[n]{a} = \sqrt[mn]{a^{m+n}}$

5. $\frac{\sqrt[m]{a}}{\sqrt[n]{a}} = \sqrt[mn]{a^{n-m}}$

2.2.6 Exercise

Exercise 1

Simplify.

1. $3x^4 - x^4 - x^3(x+2)$
2. $-12a^2 + 3a(a+1)$
3. $ax^n + 4x^n$
4. $(1-t)^2 - \frac{1}{2}(1-t)^2$
5. $a(x+t)^k - b(x+t)^k$
6. $tx^3 - 3x^2 + 2tx^3 - 4x^2$
7. $t^3 \cdot t^4 - t^5(t^2+1)$
8. $x^2 \cdot x^3 \cdot x^4$
9. $3a^k \cdot a^{k-1} \cdot a$
10. $b^n \cdot b^{2n+1}$
11. $(x+1)^{n-1} \cdot (x+1)^{n+1}$
12. $\left(\frac{x}{3}\right)^4 \cdot \left(\frac{x}{3}\right)^2$
13. $t^2 \cdot x^2 \cdot t^n \cdot x^{n-1}$
14. $a \cdot b^k \cdot a^{2n} \cdot b^{k-3}$
15. $(x-2)^n \cdot (x-2)^{1-n}$
16. $0.3^6 \cdot \left(\frac{10}{3}\right)^6$
17. $2^x \cdot \left(\frac{5}{2}\right)^x \cdot 5$
18. $2^5 \cdot \left(\frac{1}{2}\right)^4$
19. $\left(\frac{x}{4}\right)^4 \cdot 4^6$
20. $2^n \cdot \left(\frac{x}{2}\right)^n \cdot x$
21. $9 \cdot 3^{n+1}$
22. $(a-b)^9 \cdot (a-b)$
23. $\left(\frac{a-b}{c}\right)^{2k} \cdot \left(\frac{c}{a-b}\right)^{2k}$

Exercise 2

Simplify.

1. $\frac{a^6}{a^3}$
2. $\frac{x^{2n+1}}{x^n}$
3. $\frac{15e^{x+1}}{5e^x}$
4. $\frac{x^4}{x^7}$
5. $\frac{2a^{1-2n}}{4a^{n+1}}$
6. $\frac{a^4b^{4n+3}}{a^nb^{2n-1}}$
7. $\frac{81}{3^{x+3}}$
8. $\frac{(a-b)^3}{(a-b)^{n-1}}$
9. $\frac{(ab)^3}{x^2y} \cdot \frac{(xy)^2}{a^4b^2}$
10. $\frac{a^{n+1}}{a^n}$
11. $\frac{10^3}{2^3}$
12. $\frac{2.5^4}{0.5^4}$
13. $\frac{(10ab)^k}{(4b)^k}$
14. $\left(\frac{a}{b}\right)^n \cdot \frac{a}{b}$

15. $\left(\frac{-1}{a-b}\right)^3$
16. $\left(\frac{x}{2}\right)^3 : \left(\frac{x}{3}\right)$
17. $(-5^2)^3$
18. $3(c^4)^3 - 6c^{12}$
19. $(3b^2c^{n-1})^4$
20. $\left(\frac{7a^2}{49b^3}\right)^2$
21. $\left(\frac{-1}{c^3}\right)^{2n}$
22. $(3b^{n+1} \cdot c^{n-1})^2$
23. $(x^2y^3z^2)^5$
24. $(0.5e^{x+2})^2$
25. $\left(\frac{2}{x^2}\right)^5 - \left(\frac{3}{x^5}\right)^2$
26. $\left[\left(-\frac{3}{t}\right)^3\right]^4 \cdot \frac{t^9}{81}$
27. $\frac{(ab)^2}{x^3y} \cdot \frac{x^5y^2}{a^2b}$
28. $\frac{(4-12x)^3}{64}$
29. $\frac{(2x-4)^5}{(2-x)^3}$
30. $\frac{(4ab)^4}{(6a^2)^4} \cdot \frac{5}{b^4}$
31. $(a-b^2) \cdot (a-b^2)^n$

Exercise 3

Simplify.

1. $\left(\frac{1}{2}x^2\right)^5 + \frac{1}{8}(x^2)^5 + (2x^5)^2$
2. $\frac{1}{4} \cdot 2^4(2^2)^3$
3. $(3^{n+1})^2$
4. $(3x^2 - 5x)(1 - x^3) + (x^2 + 3x^4)x^3$
5. $a^{2r}b^r(a^{2r} - a^rb^{r+1} + b^{2r+2})$

Exercise 4

Simplify.

1. $-3x^3 \cdot x^2 + 5x \cdot x^4$
2. $4t^{n-4}t^3 - t \cdot t^{n-2}$
3. $2x^5y^3y - 4x^3y^2x^2y^2$
4. $\frac{4x^5+6x^4-12x^2}{2x^2}$
5. $(9 \cdot 3^n - 3^{n+1}) : 3^{n-1}$
6. $(2x+6)^2 + (x+3)^2$
7. $\frac{5a-20}{4a-16}$
8. $(3t^2 - 3t^3)^2$

Exercise 5

Factorise – write as a product by factoring out.

1. $3a^2 + 6a^3$
2. $\frac{1}{2}e^x - \frac{1}{4}e^{x+1}$
3. $a^{5b} + 3a^b$
4. $2^x + 2^{x+1}$
5. $x^4 + 2x^3$
6. $x^{n+3} - 4x^{n+2}$
7. $-6t^{n+2} + 18t^{2-n}$
8. $e^x - e^{3x}$

Exercise 6

Simplify.

1. $\frac{x^4-x^3}{x^2-x}$
2. $\frac{e^{3x}+e^{2x}}{e^{2x}}$
3. $\frac{a^7b^3-ab^7}{a^5b-a^2b^4}$
4. $\frac{32}{2^{n+5}} + \frac{2^{-n+3}}{8}$

Exercise 7

Calculate.

1. $y = \frac{1}{4}x^4 - 2tx^3 + \frac{9}{2}t^2x^2$ with $x = 3t$
2. $y = e^{x^2-t^2} + 3e^{5t-(t-x)}$ with $x = -t$
3. $y = \frac{3}{2t^2}x^4 - \frac{4}{t}x^3 + 3x^2 - 4$ with $x = \frac{1}{3}t$
4. $y = \frac{e^{3tx}+4e^3}{tx-4}$ with $x = \frac{1}{t}$
5. $y = \frac{tx^3}{2(x+t)^2}$ with $x = -3t$

Exercise 8

Simplify.

1. $\sqrt[42]{\sqrt{x}}$
2. $\sqrt[4]{x} \cdot \sqrt[3]{x}$
3. $\frac{\sqrt{3}}{\sqrt[3]{3}}$
4. $\sqrt{2.5} \cdot \sqrt{10}$

Exercise 9

Multiply and simplify.

1. $\frac{1}{4} \cdot 2^{-4} \cdot (2^2)^3$
2. $(e^x - e^{-x} + 5)e^x$
3. $2^x(2^{-1} + 2^x)$
4. $(x^4 + x^{-2})(x^3 - x^{-3})$

Exercise 10

Simplify/summarise.

1. $a^2 \cdot (a^2)^{-2} + 3a\left(\frac{1}{a}\right)^3$
2. $\frac{1}{18} \cdot (3^2)^2 + \frac{1}{2} \cdot 3^3 \cdot \left(\frac{1}{3}\right)^2$
3. $(x^2 \cdot x^{-3})^{-2} + \left(\frac{3}{x^2}\right)^{-1}$
4. $a^5 \cdot a^{-2} + 4a^2 \cdot a$
5. $\left(\frac{2}{x}\right)^3 + \left(\frac{1}{x}\right)^3$
6. $\frac{1}{e^{2x}} + 3(e^{-x})^2 - \left(\frac{2}{e^x}\right)^2$
7. $e^{-x} \cdot e^{-x+2} \cdot e^{2x-3}$
8. $6x^3 \cdot x^{-1} - 8x^4 \cdot x^{-2}$
9. $(t^7 - t^4) \cdot t^{-3}$

Exercise 11

Simplify/summarise.

1. $\frac{-2^3-2\cdot 4}{2\cdot 2^3}$
2. $\frac{(1-x)^2}{(x-1)}$
3. $\frac{e^{3x+1}}{e^{-x+2}}$
4. $\frac{1.5e^{3x}-e^x}{1.5e^{3x}}$

Exercise 12

Simplify/summarise.

1. $a^4 \cdot a^{-6} - 3a^3 \cdot a^{-5} + a^2$
2. $(a^{n+2} - 4a^n - 2a^{2-n}) \cdot \frac{a^{-2}}{2}$
3. $4x^{-4}x^7 - 0.5x^4x^{-1} + \left(\frac{1}{x^2}\right)^{1.5}$
4. $\frac{a^{n+1}}{a} + \frac{a^{2n-1}}{a^{n+2}} + (a^{n-1})^2 \cdot a^{2-n}$
5. $\frac{2^{2k}}{8} \cdot 2^{3-k} + 2 \cdot 2^{k-1}$

Exercise 13*

Simplify. (Hint: Make a case distinction.)

1. $(a-b)^n + (b-a)^n$
2. $(x-2)^n + (2x-4)^n - (2-x)^n$

2.3 Binomial Formulas

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2.3.1 Definition

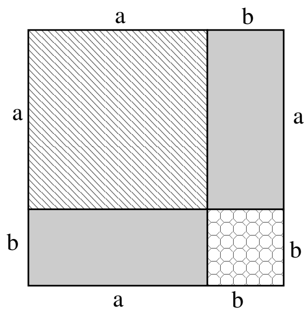
Binomial formulas are formulas for representing and solving quadratic binomials. They make it easier to multiply out expressions in brackets and allow term transformations of certain sums and differences into products. This is often the only solution strategy for simplifying fractional terms, for rooting in root expressions and logarithmic expressions.

2.3.2 Formulas

First binomial formula

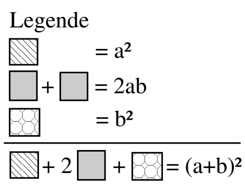
$$(a+b)^2 = a^2 + 2ab + b^2$$

The first binomial formula can be represented as shown in the following figure:



The area of a square is equal to the length of its sides squared. In the figure, the length of the sides of the square is $(a+b)$. Accordingly, the area of the entire square is $(a+b)^2$.

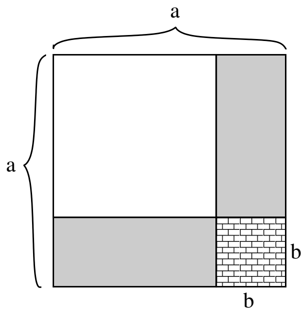
The same area is also created by combining the shaded square (area: a^2), the two grey rectangles (area: $2 \cdot ab$) and the curly square (area: b^2). The following legend is therefore obtained:



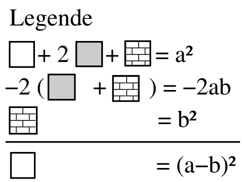
Second binomial formula

$$(a-b)^2 = a^2 - 2ab + b^2$$

The second binomial formula can be illustrated by the following figure:



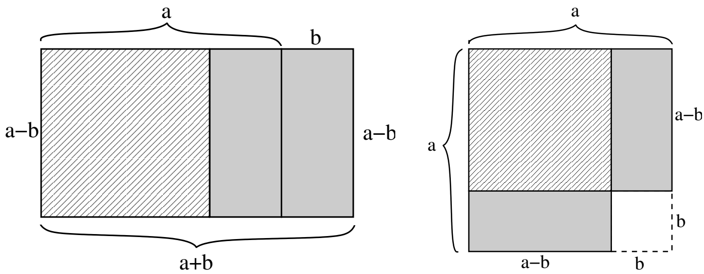
We are looking for the area of the white square: $(a-b)^2$. The entire square in the figure has an area of a^2. There are two other areas available for the calculation: the tiled square alone has an area of b^2 and, together with a grey rectangle, an area of ab. To obtain the area we are looking for, we can first remove the two grey rectangles from the entire square by subtracting $2 \cdot ab$ (i.e. $-2 \cdot ab$). However, this removes the tiled square once too often, so that it has to be added back ($+b^2$). This results in the following legend:



Third binomial formula

$$(a+b)(a-b) = a^2 - b^2$$

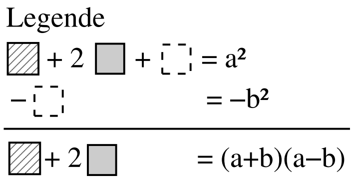
The third binomial formula can be explained with the help of the following two images:



Left image: The area of a rectangle is equal to the product of its side lengths, in this case $(a+b)$ and $(a-b)$.

Right image: We are looking for the area that consists of the shaded square and the two grey rectangles. The easiest way to get this is to (again) subtract the small white square (area: b^2) from the entire square (area: a^2).

This results in the following legend:



2.3.3 Exercises

Exercise 1

Convert the following expressions using the binomial formula.

1. $(4x + 3y^3)^2$
2. $-(x^4 - 2)^2$
3. $(x^2 - x^3)(x^2 + x^3)$
4. $(3x^2 + 2t)^2$
5. $-\frac{1}{2}(x^2 - 4)^2$
6. $\left(-\frac{1}{2}(x^2 - 4)\right)^2$
7. $x^2y^2(x^4 + 2x^2y + y^2)$

Exercise 2

Simplify. Use the binomial formulae.

1. $(x - 3)^n \cdot (x + 3)^n$
2. $\frac{(a^2-b^2)^3}{(a-b)^3}$
3. $\frac{(4-x^2)^n}{(2-x)^n}$
4. $\frac{(c-1)^{n-1}}{(c^2-1)^{n-1}}$
5. $\frac{(a^{2n}-b^{2n})^2}{(a^n-b^n)^2}$
6. $(a^3 - ab^2)(a + b)^2$
7. $\frac{[(x-y)^2]^k}{(x^2-y^2)^k}$
8. $(a + b)^4(a - b)^4(a^2 - b^2)^5$

Exercise 3

Factorise/write as a product.

1. $(3x - 6)\left(\frac{1}{4}x^2 - x + 1\right)$
2. $a^2 - 2a^3 + a^4$
3. $3a^3 - 12a^9$
4. $x^4 - a^2$
5. $3 - x^2$
6. $x^{2n} + 4x^n + 4$
7. $x^{n+2} - 6x^{n+1} + 9x^n$
8. $e^{2x} - 1$
9. $x^2e^x + 2xe^x + e^x$

Exercise 4

Simplify.

1. $\frac{a^3+2a^2b+ab^2}{(a+b)^2}$
2. $\frac{a^4-a^2b^2}{ab-a^2}$
3. $\frac{t^3+6t^2+9t}{t^2-9}$
4. $\frac{x^{2n}-10x^n+25}{x^{2n}-25}$
5. $\frac{x^6-t^2}{x^4+tx}$
6. $\frac{x^{n+3}-x^{n+1}}{x^{n+1}+x^n}$
7. $\frac{(x^2+8xy+16y^2)}{(2x-3y)^{-2}} : \frac{x^2-16y^2}{2x-3y}$
8. $\frac{4t^2-4}{t^2+2t+1}$
9. $\frac{x^{n-1}-x^n}{x^n-x^{n+2}}$
10. $\frac{2(a^2+b^2)^2}{a^5-ab^4}$
11. $\frac{x^4-x^3}{x^4-x^2}$
12. $\frac{x^3y-xy^5}{x^3y^2-x^2y^4}$
13. $\frac{am-an+bm-bn}{a^2-b^2}$

Exercise 5

Multiply and simplify.

1. $(e^x + e^{-x})^2$
2. $(a^2 - a^{-2})^2$
3. $(x^{-2} - 3x)(x^{-2} + 3x)$
4. $(2^{-x} + 2^x)(2^{-x} - 2^x)$

Exercise 6

Simplify/summarise.

1. $\frac{e^{2x}-e^{-2x}}{e^x-e^{-x}}$
2. $\left(\frac{x-y}{a-b}\right)^5 \cdot \left(\frac{x-y}{5}\right)^{-2} \cdot \frac{(a-b)^2}{(x^2-y^2)}$

2.4 Polynomial Division

Author: Gerhard Gossen Revision: Marko Rak

2.4.1 Definition of a Polynomial

A polynomial is a term of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 \quad a_n \neq 0$$

where $a_i \in \mathbb{R}$, $n \in \mathbb{N}$ and x are variables.

The degree of a polynomial ($\operatorname{grad} p(x)$) is the highest exponent of x. For example, $\operatorname{grad}(3x^2 + 2x^5 - 25x) = 5$.

2.4.2 Procedure

Given are two polynomials $p(x)$ and $q(x)$. The division $p(x) : q(x)$ yields two new polynomials:

$$p(x) : q(x) = s(x) + \frac{r(x)}{q(x)}.$$

Here $r(x)$ is the „remainder" of the division.

In the calculation, the highest terms are removed one after the other. To do this, a term $s_k = b_k x^k$ is sought that, when multiplied by the first term of q, gives the first term of p. This term is multiplied by q and subtracted from p. The resulting term p' is of a lower degree than p. s_k becomes the first term of $s(x)$ (the „result polynomial"). This procedure is repeated as long as possible, that is, as long as $\operatorname{grad} p'(x) \geq \operatorname{grad} q(x)$.

2.4.3 Example

We want to calculate $(-3 - 3x^2 + x + x^3) : (1 + x)$.

First, we arrange the polynomials according to their exponents: $(x^3 - 3x^2 + x - 3) : (x + 1)$. In the first step, x^3 is removed, so the first result term is x^2, since $x^2 \cdot x = x^3$. We then subtract $x^2(x + 1) = x^3 + x^2$.

$$\begin{array}{l}
\big(\quad x^3 - 3x^2 \; + x - 3\big) : \big(x + 1\big) = x^2 \qquad\qquad + \dfrac{\qquad}{x+1} \\
\underline{\;-x^3 \; - x^2\;} \\
\qquad\quad -4x^2 \; + x
\end{array}$$

Now we only have to calculate $(-4x^2 + x - 3) : (x + 1)$. We continue to calculate in the same way as long as possible.

$$
\begin{array}{l}
\big(\quad x^3 - 3x^2 \quad + x - 3\big) : \big(x + 1\big) = x^2 - 4x + 5 + \dfrac{}{x+1} \\
\underline{-x^3 \quad - x^2} \\
\qquad\quad - 4x^2 \quad + x \\
\qquad\quad \underline{\;\; 4x^2 + 4x} \\
\qquad\qquad\qquad 5x - 3 \\
\qquad\qquad\quad \underline{- 5x - 5} \\
\qquad\qquad\qquad\quad - 8
\end{array}
$$

We now calculate $-8 : (x + 1)$. Since $\operatorname{grad}(-8) < \operatorname{grad}(x + 1)$, the polynomial division terminates here. -8 is the „remainder“ $r(x)$ of the calculation.

$$
\begin{array}{l}
\big(\quad x^3 - 3x^2 \quad + x - 3\big) : \big(x + 1\big) = x^2 - 4x + 5 + \dfrac{-8}{x+1} \\
\underline{-x^3 \quad - x^2} \\
\qquad\quad - 4x^2 \quad + x \\
\qquad\quad \underline{\;\; 4x^2 + 4x} \\
\qquad\qquad\qquad 5x - 3 \\
\qquad\qquad\quad \underline{- 5x - 5} \\
\qquad\qquad\qquad\quad - 8
\end{array}
$$

The result of $(x^3 - 3x^2 + x - 3) : (x + 1)$ is therefore $x^2 - 4x + 5 + \frac{-8}{x+1}$. As a test, we multiply the result by $(x + 1)$.

$$
\begin{aligned}
(x^2 - 4x + 5 + \frac{-8}{x+1})(x + 1) &= x^2(x + 1) - 4x(x + 1) + 5(x + 1) + \frac{-8}{x+1}(x + 1) \\
&= (x^3 + x^2) + (-4x^2 - 4x) + (5x + 5) + (-8) \\
&= x^3 - 3x^2 + x - 3
\end{aligned}
$$

This is our original polynomial, so we have calculated correctly.

2.4.4 Further examples

$$
\begin{array}{l}
\big(\quad 4x^5 - x^4 + 2x^3 \quad + x^2 \qquad\qquad - 1\big) : \big(x^2 + 1\big) = 4x^3 - x^2 - 2x + 2 + \dfrac{2x-3}{x^2+1} \\
\underline{-4x^5 \qquad\quad - 4x^3} \\
\qquad\quad - x^4 - 2x^3 \quad + x^2 \\
\qquad\quad \underline{\;\;\; x^4 \qquad\qquad\quad + x^2} \\
\qquad\qquad\qquad - 2x^3 + 2x^2 \\
\qquad\qquad\qquad \underline{\;\;\; 2x^3 \qquad\quad + 2x} \\
\qquad\qquad\qquad\qquad\quad 2x^2 + 2x - 1 \\
\qquad\qquad\qquad\qquad \underline{- 2x^2 \qquad\quad - 2} \\
\qquad\qquad\qquad\qquad\qquad\qquad 2x - 3
\end{array}
$$

$$
\begin{array}{l}
\big(x^4 + 2x^3 - 3x^2 - 8x - 4\big) : \big(x^2 - 4\big) = x^2 + 2x + 1 \\
\underline{-x^4 \qquad\quad + 4x^2} \\
\qquad 2x^3 + x^2 - 8x \\
\qquad \underline{-2x^3 \qquad + 8x} \\
\qquad\qquad x^2 \qquad - 4 \\
\qquad\qquad \underline{-x^2 \qquad + 4} \\
\qquad\qquad\qquad\quad 0
\end{array}
$$

2.4.5 Tasks

Calculate.

1. $(x^3 + 1) : (x + 1)$
2. $(x^4 - x + 1) : (x^2 + x + 1)$
3. $(x^2 - 9) : (x + 3)$
4. $(6x^3 - 5x^2 - 36x + 35) : (3x - 7)$
5. $(x^5 - x^3 + x^2 + x - 2) : (x^2 - 1)$
6. $(3x^3 + 2x^2 + 4x + 9) : (3x + 5)$
7. $(2x^5 + 8x^4 + x^3 - x^2 + 12x + 3) : (x^2 + 4x + 1)$
8. $(x^6 - 2x^5 + 9x^4 - 8x^3 + 15x^2) : (x^2 - x + 5)$
9. $(2x^7 - x^6 + 3x^5 - \frac{1}{2}x^4 + x^3) : (2x^3 - x^2 + 2x)$
10. $(x^7 - 6x^5 + x^4 - 11x^2 - 3x + 1) : (x^3 + 2)$
11. $(3x^5 + 6x^4 + \frac{11}{3}x^3 + 4x^2 + \frac{20}{3}x) : (3x^4 + x^3 + 4x)$

3 Quadratic Equations

Author: Marc Mittner
R̈evision: Marko Rak, Julia Hempel, Johannes Jendersie

3.1 Definition

A quadratic equation is an equation that can be transformed into the form

$$ax^2 + bx + c = 0$$

with $a, b, c \in \mathbb{R}$.

A quadratic equation is in normal form if $a = 1$, i.e.

$$x^2 + px + q = 0 \quad \text{with}$$

$$\begin{aligned} p &= \frac{b}{a} \quad \text{and} \\ q &= \frac{c}{a} \end{aligned}$$

3.2 Solving quadratic equations

Every quadratic equation of the form $x^2 + px + q = 0$ has either no, one or two real solutions; in the case that the resulting parabola does not touch, touches or intersects the x axis.

3.2.1 p-q formula

Every quadratic equation in normal form ($x^2 + px + q = 0$) with $p^2 \geq 4q$ can be solved using the p-q formula. The derivation is carried out using the quadratic complement:

$$x^2 + px + q = 0$$

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

3.2.2 Midnight formula/Quadratic Formula

Any quadratic equation ($ax^2 + bx + c = 0$) with $a \neq 0$ can be solved using the midnight formula:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3.2.3 Theorem of the zero product

A product is equal to zero if and only if one of its factors is equal to zero. If an equation can be brought into the form $x^k \cdot (ax^2 + bx + c) = 0$, then according to the theorem of the zero product the equation has the solutions $x_{1,2,..,k} = 0$ and the solutions x_{k+1} and x_{k+2} can be solved using the midnight formula.

3.2.4 Substitution

If an equation has the form $ax^{2k} + bx^k + c = 0$, then x^k can be substituted by a variable u:

$$au^2 + bu + c = 0$$

This equation can then be solved as a quadratic equation. The results u_1 and u_2 are then:

$$\begin{aligned} u_1 &= x^k \\ x_{1,2} &= \sqrt[k]{u_1} \end{aligned} \qquad \begin{aligned} u_2 &= x^k \\ x_{3,4} &= \sqrt[k]{u_2} \end{aligned}$$

The number of solutions is as follows:

- no solution if $u < 0$ and k is even
- one solution if $-\infty < u < \infty$ and k is odd or $u = 0$ and k is even.
- two solutions if $u > 0$ and k is even

3.3 Examples

1. $3x^2 + 3x - 36 = 0$ Factor out:
$$\begin{aligned} 3(x^2 + x - 12) &= 0 \\ x^2 + x - 12 &= 0 \end{aligned}$$
Solve with the p-q formula ($p = 1$, $q = -12$):
$$\begin{aligned} x_{1,2} &= -\tfrac{1}{2} \pm \sqrt{\left(\tfrac{1}{2}\right)^2 + 12} \\ &= -\tfrac{1}{2} \pm \sqrt{\tfrac{49}{4}} \\ &= -\tfrac{1}{2} \pm \tfrac{7}{2} \\ x_1 &= 3 \\ x_2 &= -4 \end{aligned}$$
Factorised representation:
$$3(x-3)(x+4) = 0$$

2. $x^7 + 19x^4 - 216x = 0$

   Factorising:
   $$\begin{aligned} x(x^6 + 19x^3 - 216) &= 0 \\ x_1 &= 0 \end{aligned}$$
   Substitution of $x^3 = u$:
   $$u^2 + 19u - 216 = 0$$

   Solving with p-q formula ($p = 19$, $q = -216$):
   $$\begin{aligned} u_{1,2} &= -\tfrac{19}{2} \pm \sqrt{\left(\tfrac{19}{2}\right)^2 + 216} \\ &= -\tfrac{19}{2} \pm \sqrt{\tfrac{361}{4} + \tfrac{864}{4}} \\ &= -\tfrac{19}{2} \pm \sqrt{\tfrac{1225}{4}} \\ &= -\tfrac{19}{2} \pm \tfrac{35}{2} \\ u_1 &= 8 \\ u_2 &= -27 \end{aligned}$$
   Resubstitution:
   $x^3 = u_1$ yields the solutions
   $x^3 = 8$
   $x_2 = 2$

   $x^3 = u_2$ yields the solutions
   $x^3 = -27$
   $x_3 = -3$

3.4 Tasks

For all tasks, the following applies: $x, y, z \in \mathbb{R}$ are variables and $a, b, c \in \mathbb{R}$ are fixed parameters.

Solve the following equations:

1. $$x^2 - x - 2 = 0$$
2. $$4x^2 + 16x - 84 = 0$$
3. $$\frac{1}{2}x^2 + 3x + 4 = 0$$
4. $$4x^2 + 48x + 144 = 0$$
5. $$(x - \sqrt{157})^2 = 0$$
6. $$\frac{7}{3}x^3 + \frac{49}{3}x^2 + 35x + 21 = 0$$

7. $\frac{7}{4}x^2 + 7x = -7$

8. $|x^2| = 4$

9. $|x|^2 = 4$

10. $|x^2 - 4| = 2$

11. $x^2 = x + 12$

12. $3x^2 + 4x + 1 = 0$

13. $x^5 - 25x^3 + 144x = 0$

14. $(x - \pi)(x + \pi) = 0$

15. $\frac{x^3 - 2x^2}{x - 2} + \frac{2x^2 + 4x}{x + 2} = 1$

16. $x^4 - 14x^3 + 59x^2 - 70x = 0$

17. $3x^7 - 42x^5 + 147x^3 = 0$

18. $x^{12} = 4096$

19. $x^4 + 4x^3 + 6x^2 + 4x + 1 = 0$

20. $(\sqrt{2}x + 2\sqrt{2})^2 = 0$

21. $2ax^2 - 12ax + 18a = 0$

22. $\frac{1}{x^2} + 1 = 2$

23. $\frac{4}{x} + x = 4$

4 Linear systems of equations

Author: Marko Rak

4.1 Definition

A *linear system of equations* is a set of m linear equations containing n unknowns. In general, such a system of equations can always be represented in the following form:

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\
\vdots & & & & & & \ddots & & & & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}$$

4.2 Linear dependence

A linear equation of the above form is *linearly dependent* if it can be represented by the other equations of the system and by multiplication by a constant c_i.

$$\begin{array}{cccccccccccc}
 & a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & - & b_1 & & \\
= & (a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & - & b_2) & c_2 \\
+ & (a_{31}x_1 & + & a_{32}x_2 & + & \cdots & + & a_{3n}x_n & - & b_3) & c_3 \\
\vdots & & & & & \ddots & & & & & \vdots \\
+ & (a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & - & b_m) & c_n
\end{array}$$

Otherwise, it is *linearly independent* of the other equations in the system.

4.3 Solvability

Whether a linear system of equations is solvable and how many solutions it has depends on the case. One of the following cases always occurs

1. The system of equations has no solution.

$$\begin{array}{ccc} x_1 & = & 1 \\ x_1 & = & -1 \end{array}$$

2. The system of equations has exactly one solution.

$$\begin{array}{ccc} x_1 & = & 1 \\ x_1 + x_2 & = & -1 \end{array}$$

The system of equations has several (usually an infinite number of) solutions.

$$x_1 - x_1 = 0$$

Criteria for the solvability and the assignment of a linear system of equations to one of these cases would anticipate the lecture content and will therefore not be explained in detail here. In general, however, it can be said that if a linear system of equations has more unknowns than linearly independent equations, it has several solutions.

4.4 Solving methods

In addition to the already known solving methods such as the equalisation method, the substitution method, etc., there are also other systematic methods. These include, among others, the *Gauss method* (also known as the *Gauss algorithm*), which uses a simplified equation system representation to create a diagonal or triangular form. This speeds up the process of finding solutions.

4.4.1 Simplified representation

A general linear system of equations

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ \vdots & & & & & & \ddots & & & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & a_{m3}x_3 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

can be simplified as follows:

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & \cdots & x_n & \\ \hline a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & & & \ddots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}$$

Now the unknowns, since they are the same in all the equations of the system, are only shown in the table header. If certain unknowns do not appear in equations of the system, they are listed in this table with the factor 0. The equals sign is now represented by the separation before the last column. The addition operators are deliberately omitted and the subtraction is considered as addition with a negative operand. Elementary transformations do not change the solution of the linear system of equations. Elementary transformations are understood to mean:

1. swapping columns or rows

2. multiplying a row by a constant

3. adding a multiple of one row to another

4.4.2 Diagonal form/reading the solution

The *diagonal form* of the above general linear system of equations looks like this:

$$
\begin{array}{ccccccc|c}
x_1 & x_2 & x_3 & x_4 & \cdots & x_{n-1} & x_n & \\
\hline
a_{11} & a_{12} & a_{13} & a_{14} & \cdots & a_{1(n-1)} & a_{1n} & b_1 \\
0 & a^*_{22} & a^*_{23} & a^*_{24} & \cdots & a^*_{2(n-1)} & a^*_{2n} & b^*_2 \\
0 & 0 & a^*_{33} & a^*_{34} & \cdots & a^*_{3(n-1)} & a^*_{3n} & b^*_3 \\
0 & 0 & 0 & a^*_{44} & \cdots & a^*_{4(n-1)} & a^*_{3n} & b^*_3 \\
\vdots & & & & \ddots & & & \vdots \\
0 & 0 & 0 & 0 & \cdots & a^*_{(m-1)(n-1)} & a^*_{(m-1)n} & b^*_{m-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & a^*_{mn} & b^*_m
\end{array}
$$

This scheme makes it relatively easy to find the solutions to the linear system of equations. You start at the bottom and work your way up line by line. With each new line, you can determine another unknown.
From the last line

$$a^*_{mn} x_n = b^*_m$$

we get

$$x_n = \frac{b^*_m}{a^*_{mn}}.$$

Now we insert x_n into the second last line

$$a^*_{(m-1)(n-1)} x_{n-1} + a^*_{(m-1)n} x_n = b^*_{m-1}$$

and rearranged to x_{n-1}

$$x_{n-1} = \frac{b^*_{m-1} - \frac{a^*_{(m-1)n}}{a^*_{mn}} b^*_m}{a^*_{(m-1)(n-1)}}$$

This is continued line by line in ascending order until the first equation is reached, so that all unknowns can be determined if a solution exists.

4.4.3 Gauss algorithm

The *Gauss algorithm* mentioned above is used to create the diagonal form from any linear system of equations. To do this, the simplified representation is used and the triangular form is created step by step using elementary transformations. In each step, we select one equation and add a multiple of it to each other equation to create a column with as many zeros as possible.
The initial situation is as follows:

$$\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & \cdots & x_n & \\
\hline
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\
\vdots & & & \ddots & & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m
\end{array}$$

We select the first equation and add a multiple of it to the other equations to create zeros in the first column.

$$\begin{array}{ccccc|ccccc}
x_1 & x_2 & x_3 & \cdots & x_n & & & & & \\
\hline
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 & \cdot(-\frac{a_{21}}{a_{11}}) & \cdot(-\frac{a_{31}}{a_{11}}) & \cdots & \cdot(-\frac{a_{m1}}{a_{11}}) \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 & \hookleftarrow & & & \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 & & \hookleftarrow & & \\
\vdots & & & \ddots & & \vdots & & & \ddots & \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m & & & & \hookleftarrow
\end{array}$$

This results in the following table after the first step:

$$\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & \cdots & x_n & \\
\hline
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\
0 & a'_{22} & a'_{23} & \cdots & a'_{2n} & b'_2 \\
0 & a'_{32} & a'_{33} & \cdots & a'_{3n} & b'_3 \\
\vdots & & & \ddots & & \vdots \\
0 & a'_{m2} & a'_{m3} & \cdots & a'_{mn} & b'_m
\end{array}$$

Now we select the second equation and add a multiple of it to each subsequent equation to create zeros in the second column as well.

$$\begin{array}{ccccc|cccc}
x_1 & x_2 & x_3 & \cdots & x_n & & & & \\
\hline
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 & & & \\
0 & a'_{22} & a'_{23} & \cdots & a'_{2n} & b'_2 & \cdot(-\frac{a'_{32}}{a'_{22}}) & \cdots & \cdot(-\frac{a'_{m2}}{a'_{22}}) \\
0 & a'_{32} & a'_{33} & \cdots & a'_{3n} & b'_3 & \hookleftarrow & & \\
\vdots & & & \ddots & & \vdots & & \ddots & \\
0 & a'_{m2} & a'_{m3} & \cdots & a'_{mn} & b'_m & & & \hookleftarrow
\end{array}$$

Which brings us to the following table after the second step:

$$
\begin{array}{ccccc|c}
x_1 & x_2 & x_3 & \cdots & x_n & \\
\hline
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\
0 & a'_{22} & a'_{23} & \cdots & a'_{2n} & b'_2 \\
0 & 0 & a''_{33} & \cdots & a''_{3n} & b''_3 \\
\vdots & & & \ddots & & \vdots \\
0 & 0 & a''_{m3} & \cdots & a''_{mn} & b''_m
\end{array}
$$

This process is repeated until the desired diagonal form has been created and the generated scheme can be solved as described above.

$$
\begin{array}{cccccc|c}
x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n & \\
\hline
a_{11} & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} & b_1 \\
0 & a^*_{22} & a^*_{23} & \cdots & a^*_{2(n-1)} & a^*_{2n} & b^*_2 \\
0 & 0 & a^*_{33} & \cdots & a^*_{3(n-1)} & a^*_{3n} & b^*_3 \\
\vdots & & & \ddots & & & \vdots \\
0 & 0 & 0 & \cdots & 0 & a^*_{mn} & b^*_m
\end{array}
$$

4.5 Examples

For all examples, $x_i \in \mathbb{R}$

1. A possible task could be to solve the following system of equations:

$$
\begin{array}{rcrcrcr}
2x_1 & - & 5x_2 & + & 3x_3 & = & 3 \\
4x_1 & - & 12x_2 & + & 8x_3 & = & 4 \\
3x_1 & + & x_2 & - & 2x_3 & = & 9
\end{array}
$$

This system of equations can be simplified to the following form:

$$
\begin{array}{ccc|c}
x_1 & x_2 & x_3 & \\
\hline
2 & -5 & 3 & 3 \\
4 & -12 & 8 & 4 \\
3 & 1 & -2 & 9
\end{array}
$$

The step-by-step transformation:

x_1	x_2	x_3			
2	-5	3	3	$\cdot(-2)$	$\cdot(-\frac{3}{2})$
4	-12	8	4	$\hookleftarrow$	
3	1	-2	9		$\hookleftarrow$
2	-5	3	3		
0	-2	2	-2	$\cdot(\frac{17}{4})$	
0	$\frac{17}{2}$	$-\frac{13}{2}$	$\frac{9}{2}$	$\hookleftarrow$	
2	-5	3	3		
0	-2	2	-2		
0	0	2	-4		

Thus, the last line directly gives $2x_3 = -4$ and therefore $x_3 = -2$. This result can in turn be inserted into the equation of the second line ($-2x_2 + 2x_3 = -2$). The resulting equation $-2x_2 + 4 = -2$ has the solution $x_2 = -1$. The results for x_2 and x_3 can now be inserted into the equation in the first line ($2x_1 - 5x_2 + 3x_3 = 3$) and, after rearranging, we obtain $x_1 = 2$. The last three equations therefore give the following result:

$$\begin{aligned} x_3 &= -2 \\ x_2 &= -1 \\ x_1 &= 2 \end{aligned}$$

The linear system of equations therefore has exactly one solution.

2. The initial situation is as follows:

$$\begin{array}{rcrcrcr} 3x_1 & - & 1x_2 & + & 2x_3 & = & 1 \\ 7x_1 & - & 4x_2 & - & 1x_3 & = & -2 \\ -x_1 & - & 3x_2 & - & 12x_3 & = & -5 \end{array}$$

and can be simplified to:

x_1	x_2	x_3	
3	-1	2	1
7	-4	-1	-2
-1	-3	-12	-5

Now the diagonal form is generated step by step.

To save space, all steps can be carried out in a table.

x_1	x_2	x_3			
3	-1	2	1	$\cdot(-\frac{7}{3})$	$\cdot(\frac{1}{3})$
7	-4	-1	-2	$\hookleftarrow$	
-1	-3	-12	-5		$\hookleftarrow$
3	-1	2	1		
0	$-\frac{5}{3}$	$-\frac{17}{3}$	$-\frac{13}{3}$	$\cdot(-2)$	
0	$-\frac{10}{3}$	$-\frac{34}{3}$	$-\frac{14}{3}$	$\hookleftarrow$	
3	-1	2	1		
0	$-\frac{5}{3}$	$-\frac{17}{3}$	$-\frac{13}{3}$		
0	0	0	4		

Once the diagonal form has been created, the result can be easily derived as described above. In this example, a contradiction arises in the last equation:

$$0x_1 + 0x_2 + 0x_3 = 4.$$

Thus, the linear system of equations has no solution.

3. One last example in a nutshell.

x_1	x_2	x_3			
1	-2	3	4	$\cdot(-3)$	$\cdot(-2)$
3	1	-5	5	$\hookleftarrow$	
2	-3	4	7		$\hookleftarrow$
1	-2	3	4		
0	7	-14	-7	$\cdot(-\frac{1}{7})$	
0	1	-2	-1	$\hookleftarrow$	
1	-2	3	4		
0	7	-14	-7		
0	0	0	0		

A zero row has been created, which occurs when two equations are linearly dependent. Consequently, the linear system of equations now only has 2 (linearly independent) equations and 3 unknowns. We can choose any variable, which leads to an infinite number of solutions for this linear system of equations. We therefore set

$$x_3 = t, t \in \mathbb{R}$$

and now solve the other unknowns in dependence on t .

$$\begin{array}{ccccc} x_2 & = & -1 & + & 2t \\ x_1 & = & 2 & + & t \end{array}$$

4.6 Tasks

For all tasks, $x_i \in \mathbb{R}$ and $a, b \in \mathbb{R}$ are fixed.

4.6.1 Simple systems of equations

Determine the solutions to the following systems of equations.

1. $$\begin{array}{rcrcrcr} 7x_1 & + & 8x_2 & + & 5x_3 & = & 3 \\ 3x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 18x_1 & + & 21x_2 & + & 13x_3 & = & 8 \end{array}$$

2. $$\begin{array}{rcrcrcr} x_1 & + & 5x_2 & + & 2x_3 & = & 3 \\ 2x_1 & - & 2x_2 & + & 4x_3 & = & 5 \\ x_1 & + & x_2 & + & 2x_3 & = & 1 \end{array}$$

3. $$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & 3x_3 & + & 4x_4 & = & -3 \\ 2x_1 & + & 3x_2 & + & 11x_3 & + & 5x_4 & = & 2 \\ 2x_1 & + & x_2 & + & 3x_3 & + & 2x_4 & = & -3 \\ x_1 & + & x_2 & + & 5x_3 & + & 2x_4 & = & 1 \end{array}$$

4. $$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & -4 \\ 5x_1 & - & x_2 & + & x_3 & = & 0 \\ 7x_1 & + & 3x_2 & + & 7x_3 & = & -8 \\ 2x_1 & + & 3x_2 & - & x_3 & = & 11 \end{array}$$

5. $$\begin{array}{rcrcrcrcrcr} -x_1 & + & x_2 & + & x_3 & & & - & x_5 & = & 0 \\ x_1 & - & x_2 & - & 3x_3 & + & 2x_4 & - & x_5 & = & 2 \\ & & 3x_2 & - & x_3 & - & 5x_4 & - & 7x_5 & = & 9 \\ 3x_1 & - & 3x_2 & - & 5x_3 & + & 2x_4 & + & 5x_5 & = & 2 \end{array}$$

6. $$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & & & = & -7 \\ 2x_1 & - & x_2 & + & 2x_3 & + & 7x_4 & = & -3 \\ -2x_1 & + & x_2 & + & 3x_3 & + & 3x_4 & = & 8 \\ x_1 & + & 4x_2 & + & 5x_3 & - & 2x_4 & = & 7 \end{array}$$

7. $$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 4 \\ x_1 & + & 2x_2 & + & x_3 & = & 13 \\ 4x_1 & + & 5x_2 & + & 4x_3 & = & 43 \\ 2x_1 & + & 4x_2 & + & 2x_3 & = & 26 \end{array}$$

4.6.2 Parametrised systems of equations*

Determine the solutions of the following systems of equations in terms of a and b.

1.
$$\begin{array}{rcrcrcl} 2x_1 & - & x_2 & + & 4x_3 & = & 0 \\ x_1 & + & 3x_2 & - & x_3 & = & 0 \\ 7x_1 & + & 7x_2 & + & (4-a)x_3 & = & 0 \end{array}$$

2.
$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 0 \\ x_1 & + & ax_2 & + & x_3 & = & 4 \\ ax_1 & + & 3x_2 & + & ax_3 & = & -2 \end{array}$$

3.
$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & + & 3x_3 & = & 4 \\ 2x_1 & + & x_2 & + & x_3 & = & -2 \\ x_1 & + & ax_2 & + & 2x_3 & = & b \end{array}$$

5 Absolute value and inequalities

5.1 Absolute value

Author: Marc Mittner
'Revision: Christian Braune

5.1.1 Definition

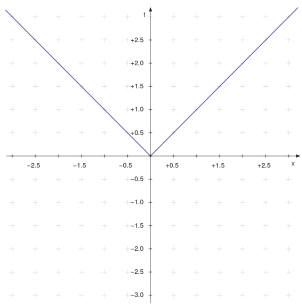
For a real number x, the absolute value is defined as:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

5.1.2 The absolute value function

The graph of the absolute value function $f(x) = |x|$ is:

- symmetric with respect to the y-axis
- $y \geq 0$ for all values $x \in \mathbb{R}$.



5.1.3 Rules for calculating the absolute value

1. $|-a| = |a|$
2. $|a| \geq 0; |a| = 0 \Leftrightarrow a = 0$
3. $|a \cdot b| = |a| \cdot |b|$
4. $|\frac{a}{b}| = \frac{|a|}{|b|}$ for b$\neq 0$
5. $|a^n| = |a|^n$ for n$\in \mathbb{N}$
6. $|a + b| \leq |a| + |b|$ (so-called triangle inequality)

5.1.4 Examples

Equations with absolute values are solved by case distinction.

1. $|x-1| = 3$

   Case distinction

   Case 1:

   $$\begin{aligned} +(x-1) &= 3 \\ x &= 4 \end{aligned}$$

   Case 2:

   $$\begin{aligned} -(x-1) &= 3 \\ x &= -2 \end{aligned}$$

2. $(x+3)^2 = 4 \to |x+3| = 2$

   Case distinction

   1st case:

   $$\begin{aligned} +(x+3) &= 2 \\ \to x &= -1 \end{aligned}$$

   2nd case:

   $$\begin{aligned} -(x+3) &= 2 \\ \to -x-3 &= 2 \\ \to x &= -5 \end{aligned}$$

5.1.5 Tasks

Solve the following equations:

1. $|x| = 7$
2. $|x+5| = 10$
3. $|2x-3| = 1$
4. $|2x-4| = 6x+36$

5.2 Inequalities

Author: Marc Mittner
‘Revision: Christian Braune

5.2.1 Definition

An inequality represents an order of two mathematical objects. Inequalities are distinguished according to the number of variables and the power in which the variables occur. The solution method varies depending on the type of inequality.

5.2.2 "Equivalence transformation of inequalities

The following operations are permitted for transforming inequalities:

- Addition of a number $a \in \mathbb{R}$ on both sides.
- Subtraction of a number $a \in \mathbb{R}$ on both sides.
- Multiplication/division by a number $a \in \mathbb{R}, a > 0$ on both sides.
- Multiplication/division by a number $a \in \mathbb{R}, a < 0$ on both sides. Note that the sign of the absolute value is reversed!
- When taking the square root, it is important to note that the inequality breaks down into two parts:

$$\begin{aligned} x^2 < a^2 \Leftrightarrow \\ -a < x < a \Leftrightarrow \\ |x| < a \end{aligned}$$

"Change of the order signs when multiplying/dividing by a number $a < 0$:

- $<$ becomes $>$, $>$ becomes $<$
- $\leq$ becomes $\geq$, $\geq$ becomes $\leq$ The signs “$=$‘ and “‘$\neq$”’ are retained.

The following transformations are not generally allowed:

- multiplication by 0 on both sides
- division by 0 on both sides
- squaring on both sides

5.2.3 Linear inequalities

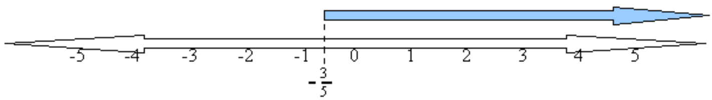
A linear inequality is an inequality in which the variable is only contained in the first power. Every linear inequality can be put into one of these three forms:

$$ax + b > c \quad \text{or} \quad ax + b \geq c \quad \text{or} \quad ax + b \neq c$$

To solve a linear inequality, the variable is isolated by rearranging the terms:
Example:

$$\begin{aligned} 3 - 4x - 13 + 2x - 3x + 12 &\leq 5 && \text{Combine} \\ -5x + 2 &\leq 5 && \mid -2 \\ -5x &\leq 3 && \mid \div(-5) \\ x &\geq -\frac{3}{5} \end{aligned}$$

Graphical representation of the solution area:



General:

$$\begin{aligned} ax + b &\leq c && \mid -b \\ ax &\leq c - b && \mid \div a \ (a > 0) \\ x &\leq \frac{c-b}{a} \end{aligned}$$

5.2.4 Inequalities with several variables

Inequalities with several variables have a multidimensional solution space instead of a one-dimensional one. The dimension increases with the number of variables. Thus, an inequality system with two variables has a solution in $\mathbb{R}^2$ (i.e. in a plane) and inequality systems with n variables have a solution in $\mathbb{R}^n$.

Example: inequality with 2 variables

$$\begin{aligned} 2x^2 + 3 - y - 2 &> 2 \\ 2 - \frac{1}{2}x &< \frac{1}{2}y + 2 \end{aligned}$$

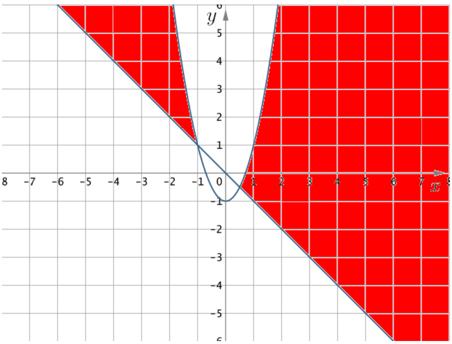
To solve the system of inequalities, one variable is isolated first.

$$y < 2x^2 - 1$$
$$y > -x$$

This results in:

$$2x^2 - 1 > y > -x$$

Graphical representation of the solution set:



The marked area represents the solution set.
The points on the functions themselves are not included in the solution set.
There is no solution for the range $-1 \leq x \leq \frac{1}{2}$. For all other values of x, all points for which the condition

$$2x^2 - 1 > y > -x$$

is satisfied are included in the solution set.

5.2.5 Tasks

Inequalities with one variable

Solve the following systems of inequalities analytically:

1. $$\frac{1}{2}x^2 - 1 > 0$$

2. $$(x+1)(x-1) \leq 0$$
$$\sqrt{x} \geq 1$$

3. Give the solution set of the system of inequalities in dependence on a.

$$ax^2 > 0$$
$$\frac{1}{2}x + 1 > 0$$

4. Give the solution set of the system of inequalities in dependence on a.

$$\begin{aligned} x^2 + a &> 0 \\ \frac{1}{2}x + 1 &> 0 \end{aligned}$$

5. Give the solution set of the system of inequalities in dependence on a.

$$\begin{aligned} -x^2 + a &< 0 \\ x + a &< 0 \end{aligned}$$

6. $$\begin{aligned} (x-1)^2 - 4 &< 0 \\ -(x+1)^2 + 4 &> 0 \end{aligned}$$

7. $$\begin{aligned} \sqrt{(x-1)} &\geq 0 \\ -\frac{1}{4x} + 4 &< 0 \end{aligned}$$

8. $$\begin{aligned} x^4 - 16 &\leq 0 \\ x^3 + 1 &\geq 0 \end{aligned}$$

Inequalities with several variables

Graphically solve the following systems of inequalities:

1. $$\begin{aligned} -x^2 + 5 &< y \\ x(x-3)^2 &> y \\ -x - 2 &> y \end{aligned}$$

2. $$\begin{aligned} 3x^2 - 3x - 10 &< -4 + y \\ y &\leq \frac{1}{2} \end{aligned}$$

3. $$\begin{aligned} \frac{1}{2}x^2 - 3x &\leq y \\ y &\leq -x \\ 17x^3 - \frac{1}{2} &= y \end{aligned}$$

4. $$\begin{aligned} y + \sqrt{\frac{x^3 + x^2 - x - 1}{x - 1}} &> 0 \\ \frac{2}{20}x - \frac{1}{3}y + \frac{3}{12} &< 0 \end{aligned}$$

5.
$$\begin{aligned}
\frac{1}{2}x - 2 &< y\\
\frac{1}{2}x + 2 &> y\\
2x - 4 &< y\\
2x + 4 &> y\\
-\frac{1}{2}x - 2 &< y\\
-\frac{1}{2}x + 2 &> y\\
-2x - 4 &< y\\
-2x + 4 &> y
\end{aligned}$$

6.
$$\begin{aligned}
((\sin x) + \frac{1}{2})^2 - \frac{3}{4} - y - (\sin x)^2 &> 0\\
\cos(x + \frac{\pi}{2}) + \frac{1}{2} &< y
\end{aligned}$$

7.
$$\begin{aligned}
\left|\frac{1}{x}\right| &> y\\
-\frac{1 + 7x^2}{x^2 y} &> -\frac{y+7}{y}\\
|x| + y &< 5
\end{aligned}$$

8. Calculate the area of the solution set for the inequality:

$$\begin{aligned}
(2y-3)^2 + (3y+2)^2 + y - 10 &\geq \left|\frac{4x + 4(\frac{1}{2}x - \frac{3}{2})^2 - 9}{x}\right| + 13y^2\\
y &\leq -1
\end{aligned}$$

9. F"or which a is the area of the solution set equal to 2?

$$\begin{aligned}
y &\geq 2\\
-|x| + a &\leq y
\end{aligned}$$

6 Complete induction

Author: Katja Matthes ‘Revision: Sebastian Nielebock

6.1 Principle

Complete induction is a mathematical method of proof. The aim of complete induction is to prove the validity of a statement for all natural numbers $n \geq \in \mathbb{N}_0$ (induction start).

1. **Induction start**
   It is shown that the statement is true for the natural number $n_0 = 1$ (or also $n_0 = 0, 2, 3, ...$).

2. **Induction step**
   - Induction assumption: It is assumed that the statement is true for a fixed natural number n.
   - Induction statement: It is claimed that the statement also applies to the following natural number $n+1$ under the assumption.
   - Induction proof: The induction statement is proved using the induction assumption.

3. **Conclusion**
   The combination of the induction start and induction step shows that the statement actually applies to all natural numbers $n \geq n_0$.

6.2 Insert: The summation sign

Many induction problems are formulated using summation signs. In order to be able to better carry out some of the proofs, it is necessary to know a few rules.

6.2.1 General

The summation sign represents a shortcut for addition:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

6.2.2 Factoring out the last index

A very useful rule for induction proofs is to factor out the last index. This makes it easy to show the induction step for many proofs:

$$\sum_{i=l}^{n+1} a_i = \left(\sum_{i=l}^{n} a_i\right) + a_{n+1}$$

6.2.3 Associative law

$$\sum_{i=l}^{m-1} a_i + \sum_{i=m}^{n} a_i = \sum_{i=l}^{n} a_i$$

6.3 Example problem

Show: For all $n \in \mathbb{N}$, the equation

$$\sum_{k=1}^{n} 2^k = 2(2^n - 1)$$

6.3.1 Induction start

We show that the statement is true for $n_0 = 1$.

$$\sum_{k=1}^{1} 2^k = 2^1 = 2 = 2(2^1 - 1)$$

(true statement)

6.3.2 Induction step

Induction Assumption: We assume that the assumption is valid for a fixed $n \in \mathbb{N}$

$$\sum_{k=1}^{n} 2^k = 2(2^n - 1)$$

Induction hypothesis: We claim that the statement also applies to the next number $n + 1$ (i.e., $n \mapsto n + 1$):

$$\sum_{k=1}^{n+1} 2^k = 2(2^{n+1} - 1)$$

Induction proof: Using the induction hypothesis, the left side of the statement is transformed into its right side.

$$\begin{aligned}
\sum_{k=1}^{n+1} 2^k &= \sum_{k=1}^{n} 2^k + 2^{n+1} && |\text{according to induction hypothesis} \\
&= 2(2^n - 1) + 2^{n+1} && |\text{power laws} \\
&= 2(2^n - 1) + 2 \cdot 2^n && |\text{factor out 2} \\
&= 2(2^n - 1 + 2^n) && |\text{summarise} \\
&= 2(2 \cdot 2^n - 1) && |\text{power laws} \\
&= 2(2^{n+1} - 1)
\end{aligned}$$

6.3.3 Induction conclusion

Thus, by the principle of complete induction, the induction claim and thus also the statement:

$$\sum_{k=1}^{n} 2^k = 2(2^n - 1)$$

for all $n \in \mathbb{N}$ is proven.

qed.

6.4 Problems

Equations

1. Proof: For all $n \in \mathbb{N}$, the sum formula

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

2. Proof: For all $n \in \mathbb{N}$, the sum formula

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

3. Proof: The sum of the first n even natural numbers is equal to $n^2 + n$, i.e.

$$\sum_{k=1}^{n} 2k = n^2 + n$$

4. Proof: For all $n \in \mathbb{N}$, the sum formula

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$$

5. The sum of the first n odd natural numbers $1+3+5+...+2n-1$ is to be determined. Make a conjecture and prove it by complete induction.

6. The sum of $4+8+12+...+4n$, i.e. the first n natural numbers divisible by 4, is to be determined. Make a conjecture and prove it by complete induction.

7. Proof: For all $n \in \mathbb{N}$, the sum formula (with $0 < q < 1$) applies

$$\sum_{k=0}^{n} q^k = \frac{1-q^{n+1}}{1-q}$$

8. Proof: For all $n \in \mathbb{N}$, the sum formula

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}$$

9. Proof: For all $n \in \mathbb{N}$, the sum formula

$$\sum_{k=1}^{n} \frac{k}{2^k} = 2 - \frac{n+2}{2^n}$$

10. Proof:

$$\sum_{k=0}^{n} \left(\frac{2}{3}\right)^k = 3 \cdot \left(1 - \left(\frac{2}{3}\right)^{n+1}\right)$$

Divisibility problems

1. Proof: For all natural numbers n, 8 is a divisor of $9^n - 1$

2. Proof: For all natural numbers n, 6 is a divisor of $7^n - 1$

3. Proofs: For all natural numbers n, $a-1$ is a divisor of $a^n - 1$ with $a \in \mathbb{R}$ and $a > 1$

4. Proofs that the term $n^3 + 6n^2 + 14n$ is a multiple of 3 for all natural numbers.

5. Proofs: For all natural numbers n, 3 is a divisor of $2^{2n} - 1$

6. Proof: For all natural numbers n, 6 is a divisor of $n^3 - n$

7. Proof: For all natural numbers n, $3n^2 + 9n$ is divisible by 6

Inequalities

1. Prove the **Bernoulli inequality**: For all $n \in \mathbb{N}$ and $x \geq -1$, $(1+x)^n \geq 1+nx$
2. Determine the smallest natural number n_0 for which the following inequality is true: $n^2 + 10 < 2^n$. Prove that the inequality is true for all natural numbers $n \geq n_0$.
3. Prove: For all natural numbers $n \geq 3$, $n^2 > 2n+1$
4. Proof: For all natural numbers $n \geq 5$: $2^n > n^2$
5. Proof: For all natural numbers $n \geq 2$:
$$\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$$
6. Proof: For all natural numbers $n > 2$:
$$\sum_{k=1}^{n} \frac{1}{n+k} > \frac{13}{24}$$

7 Functions

Author: Gerhard Gossen

A function f is a mapping that assigns a value from the *domain* $D(f)$ to exactly one value from the *range* $W(f)$. The usual representation is $f : X \to Y$ (say: f is a mapping from X to Y), where X is the domain ($D(f) \subseteq X$) and Y is the range ($W(f) \subseteq Y$). The domain and range are often $\mathbb{R}$ (the real numbers).
Common functions are, for example, straight lines ($f(x) = m \cdot x + n$), polynomials ($f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$), the trigonometric functions ($\sin x$, $\cos x$, $\tan x$, see section 7.1) or exponential functions (a^x, see section **??**). Figure **??** shows the graphs of some functions.

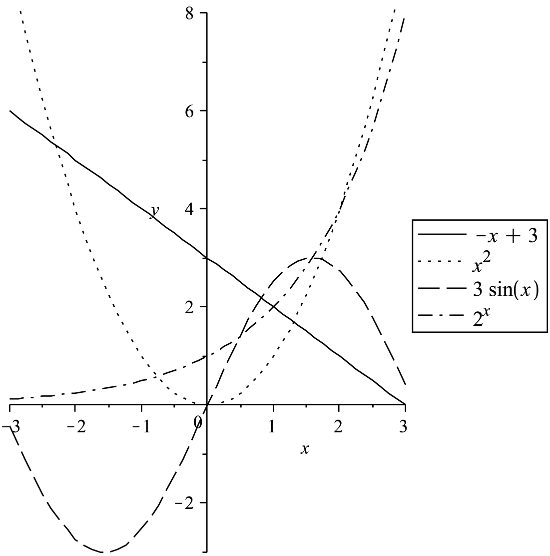


Figure 7.1: Known Functions

All the functions we will be dealing with in the preliminary course are functions with *one variable*, i.e. functions that depend on a single variable (usually x). The properties of a function can be determined using a *curve sketching* (see section 7.3). However, first we will introduce two important families of functions: the trigonometric functions (angle functions, section 7.1) and the exponential functions (section **??**).

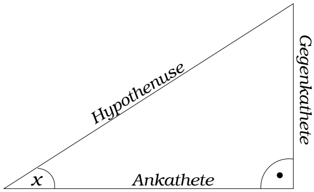
7.1 Trigonometric Functions

The trigonometric functions sin, cos, tan are the angular functions. They are defined for angles in radians (e.g. $x = \frac{\pi}{2}$). Angles in degrees (e.g. $x = 90°$) can be converted unambiguously into radians ($90° \equiv \frac{\pi}{2}$). This is why the notation $\sin 90°$ is also possible.

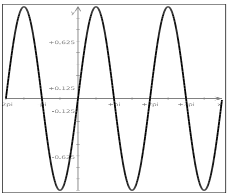
7.1.1 Definition

In a right-angled triangle, the following applies:

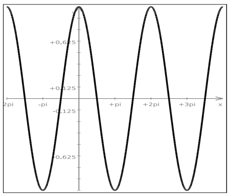
$$\begin{aligned} \sin x &= \frac{\text{opposite}}{\text{hypotenuse}} \\ \cos x &= \frac{\text{adjacent}}{\text{hypotenuse}} \\ \tan x &= \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin x}{\cos x} \end{aligned}$$



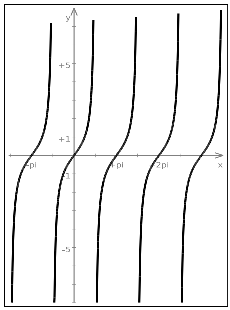
These definitions can be generalised so that the functions are defined for all real numbers (in the right-angled triangle: $0 \leq x \leq 90°$). This results in the following functions:



$\sin x$



$\cos x$



$\tan x$

We can see that the cosine is only shifted by $\frac{\pi}{2}$ relative to the sine, so the following applies:

$$\sin\left(\frac{\pi}{2} + x\right) = \cos x \qquad \cos\left(\frac{\pi}{2} - x\right) = \sin x$$

Example:

$$\begin{aligned} \cos\left(\frac{\pi}{2}\right) &= \sin\left(\frac{\pi}{2} - \frac{\pi}{2}\right) = \sin(0) = 0 \\ \sin\left(\frac{3\pi}{2}\right) &= \cos\left(\frac{\pi}{2} - \frac{3\pi}{2}\right) = \cos(-\pi) = -1 \end{aligned}$$

7.1.2 Periodicity and Symmetry

All trigonometric functions are periodic, i.e. all values repeat at regular intervals. Thus, for any integer k:

$$\sin(x + 2k\pi) = \sin x \quad \cos(x + 2k\pi) = \cos x \quad \tan(x + k\pi) = \tan x$$

(Remember: $2\pi \equiv 360°$, so $2k\pi$ is exactly k full circles).

Examples:

$$\sin\left(\frac{\pi}{2}\right) = \sin\left(\frac{5\pi}{2}\right) = \sin\left(\frac{9\pi}{2}\right) = \sin\left(\frac{13\pi}{2}\right) = \dots$$
$$\cos(\pi) = \cos(3\pi) = \cos(5\pi) = \cos(7\pi) = \dots$$
$$\tan\left(\frac{\pi}{4}\right) = \tan\left(\frac{3\pi}{4}\right) = \tan\left(\frac{5\pi}{4}\right) = \tan\left(\frac{5\pi}{4}\right) = \dots$$

The sine and cosine functions are symmetrical, the sine is symmetrical about the origin, and the cosine is symmetrical about the y axis. Together with periodicity, the following relationships follow:

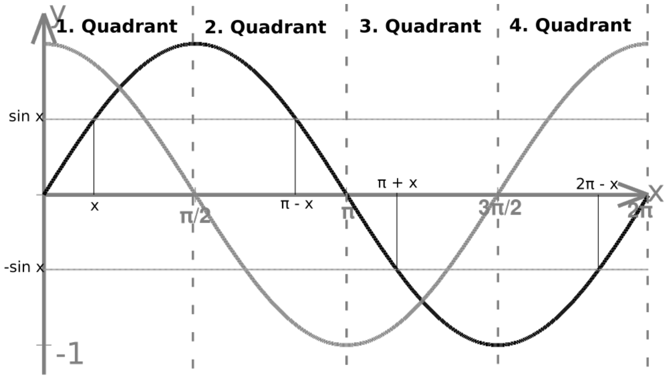
$$\begin{aligned}
\sin(\pi - x) &= \sin x & \cos(\pi - x) &= -\cos x & \tan(\pi - x) &= -\tan x \\
\sin(\pi + x) &= -\sin x & \cos(\pi + x) &= -\cos x & \tan(\pi + x) &= tanx \\
\sin(-x) &= -\sin x & \cos(-x) &= cosx & \tan(-x) &= -\tan x
\end{aligned}$$

The sign therefore depends on the quadrant in which x is located. Figure 7.2 shows this relationship.
The values for $\sin x, 0 \le x \le \frac{\pi}{2}$ are sufficient to determine all values for sin and cos. The most important values are given in the following table.

x in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
x in degrees	0	$30°$	$45°$	$60°$	$90°$
$\sin x$	$\frac{1}{2}\sqrt{0}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
$\cos x$	$\frac{1}{2}\sqrt{4}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{0}$

7.1.3 Inverse functions

The inverse functions of sin, cos and tan are arcsin, arccos and arctan (pronounced: *arcus sinus*, *arcus cosinus* and arc tangent). Since the trigonometric functions are periodic (e.g. $\sin(0) = \sin(2\pi) = 0$), there cannot be a unique inverse function (in this



$\sin x$	$+\sin x$	$-\sin x$	$-\sin x$
$\cos x$	$-\cos x$	$-\cos x$	$+\cos x$
$\tan x$	$-\tan x$	$+\tan x$	$-\tan x$

Figure 7.2: Symmetry of sine and cosine

example: is arcsin(0) equal to 0 or 2π?). Therefore, the functions are only invertible in a certain range. These ranges are:

$$
\begin{array}{cccll}
y = \sin x & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} & \Longleftrightarrow & x = \arcsin y & -1 \leq y \leq 1 \\
y = \cos x & 0 \leq x \leq \pi & \Longleftrightarrow & x = \arccos y & -1 \leq y \leq 1 \\
y = \tan x & -\frac{\pi}{2} < x < \frac{\pi}{2} & \Longleftrightarrow & x = \arctan y & y \in \mathbb{R}
\end{array}
$$

7.1.4 Trigonometric Pythagoras

For all x: $\sin^2 x + \cos^2 x = 1$. This statement is also called *trigonometric Pythagoras*. This can sometimes be used to simplify a term.

7.1.5 Addition theorems

The trigonometric functions have, among other things, these important properties:

$$
\begin{aligned}
\sin(x_1 + x_2) &= \sin x_1 \cos x_2 + \cos x_1 \sin x_2 \\
\sin(x_1 - x_2) &= \sin x_1 \cos x_2 - \cos x_1 \sin x_2 \\
\cos(x_1 + x_2) &= \cos x_1 \cos x_2 - \sin x_1 \sin x_2 \\
\cos(x_1 - x_2) &= \cos x_1 \cos x_2 + \sin x_1 \sin x_2
\end{aligned}
$$

Examples:

$$\begin{aligned}
\sin(120^\circ) &= \sin(60^\circ + 60^\circ) = \sin(\tfrac{\pi}{3} + \tfrac{\pi}{3}) && \text{(apply the addition theorem).}\\
&= \sin\tfrac{\pi}{3}\cos\tfrac{\pi}{3} + \cos\tfrac{\pi}{3}\sin\tfrac{\pi}{3}\\
&= \tfrac{1}{2}\sqrt{3}\cos\tfrac{\pi}{3} + \cos\left(\tfrac{\pi}{3}\right)\tfrac{1}{2}\sqrt{3} = \sqrt{3}\cos\tfrac{\pi}{3} && \text{(change cos to sin)}\\
&= \sqrt{3}\sin\left(\tfrac{\pi}{2} - \tfrac{\pi}{3}\right) = \sqrt{3}\sin\tfrac{\pi}{6}\\
&= \tfrac{1}{2}\sqrt{3}
\end{aligned}$$

Test of symmetry:

$$\begin{aligned}
\sin(120^\circ) &= \sin\left(\tfrac{2\pi}{3}\right) = \sin(\pi - \tfrac{\pi}{3}) \quad \text{(using 7.1.2)}\\
&= \sin(\tfrac{\pi}{3}) = \tfrac{1}{2}\sqrt{3}
\end{aligned}$$

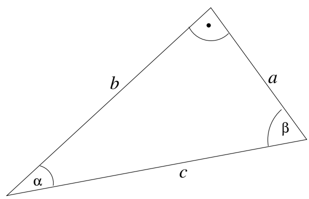
7.1.6 Tasks

1. Using the table in section 7.1.2, calculate the following values:

a) $\sin\frac{2\pi}{3}$	f) $\sin\frac{7\pi}{3}$	k) $\cos\frac{\pi}{3}$	p) $\cos\frac{4\pi}{6}$
b) $\sin\frac{5\pi}{6}$	g) $\sin\frac{29\pi}{6}$	l) $\cos\frac{\pi}{2}$	q) $\cos\frac{7\pi}{3}$
c) $\sin\pi$	h) $\sin-\frac{3\pi}{4}$	m) $\cos\frac{11\pi}{6}$	r) $\cos-\frac{11\pi}{4}$
d) $\sin\frac{3\pi}{2}$	i) $\cos\frac{\pi}{6}$	n) $\cos\frac{3\pi}{4}$	s) $\tan\frac{\pi}{6}$
e) $\sin\frac{11\pi}{6}$	j) $\cos\frac{\pi}{4}$	o) $\cos\frac{2\pi}{3}$	t) $\tan-\frac{\pi}{3}$

Calculate the missing side lengths and angles. The notations correspond to the drawing on the left.

α	β	a	b	c
			1	$\sqrt{2}$
		2		4
		$\frac{1}{2}\sqrt{3}$		$\frac{1}{2}$
		4	3	
$\frac{\pi}{6}$		1		
	$\frac{\pi}{3}$		2	



2. * Derive the following statement:

$$\sin(4\alpha) = 4(\sin\alpha\cdot\cos^3\alpha - \sin^3\alpha\cdot\cos\alpha)$$

3. * Derive the following statement:

$$\cos(2\alpha) = 2\cdot\cos^2\alpha - 1$$

4. * A sphere with a radius of 1 encloses a cube. Determine the maximum side length of the cube.

7.2 Exponential functions and logarithms

7.2.1 Exponential functions

Exponential functions are functions of the form $f(x) = a^x$, where a is a constant number > 0. These functions have some common properties:

- $f(x) > 0$, i.e. in particular the function has no zero
- $f(0) = 1$, since $a^0 = 1$ for all a

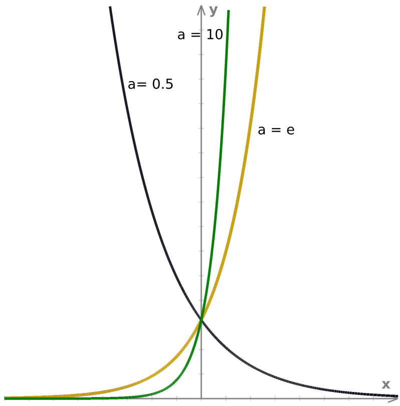


Figure 7.3: Graph of the function a^x, with $a \in \{e, 10, \frac{1}{2}\}$.

The function has the following form depending on a:

$\mathbf{a > 1}$ strictly monotonically increasing, for $x \to -\infty$ $f(x)$ tends to 0.

$\mathbf{0 < a < 1}$ strictly monotonically decreasing, for $x \to \infty$ $f(x)$ tends to 0.

$\mathbf{a = 1}$ the function yields a constant 1 ($f(x) = 1^x$).

A special exponential function is the e function $f(x) = e^x$, which can be used to describe many natural processes. $e = 2.718281828459\ldots$ is *Euler's number*. The e function has the property that $e^x = (e^x)' = (e^x)'' = (e^x)^{(n)}$, i.e. all derivatives of the function are equal to the function.
The inverse function of the exponential function is the logarithm:

$$y = a^x \Longleftrightarrow x = \log_a y.$$

7.2.2 Logarithm

The logarithm $\log_a b$ (spoken: logarithm of b to the base a) is the number c for which $a^c = b$. The logarithm is thus the inverse function of the exponential function. Important logarithms are:

Natural logarithm logarithm to the base e: $\log_e x = \ln x$

Decadic logarithm logarithm to the base 10: $\log_{10} x = \lg x$

Binary logarithm logarithm to the base 2: $\log_2 x = \mathrm{ld}\, x$

7.2.3 Logarithm function

The graph of the logarithm function (Figure 7.3) behaves similarly to the graph of the exponential function: depending on a, the graph is either strictly monotonically decreasing ($0 < a < 1$) or strictly monotonically increasing ($a > 1$). The function is only defined for positive numbers, the limit for $x \to 0$ is $\pm\infty$. The function value $\log_a(1)$ is 0, regardless of a.

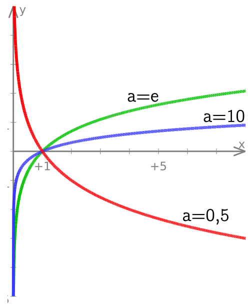


Figure 7.4: Graph of the function $\log_a x$, with $a \in \{e, 10, \frac{1}{2}\}$.

For large values of x (and $n > 0$), the following applies: $log_a(x) < n \cdot x$, so the logarithm function grows more slowly than a linear function.

7.2.4 Logarithmic laws

$$\begin{aligned}
\log_a(u \cdot v) &= \log_a u + \log_a v \\
\log_a \frac{u}{v} &= \log_a u - \log_a v \\
\log_a u^r &= r \cdot \log_a u
\end{aligned}$$

7.2.5 Base change

A logarithm to an unusual base a can be calculated by converting it to a different base b:

$$\log_a x = \frac{\log_b x}{\log_b a}, \text{ e.g. } \log_a x = \frac{\ln x}{\ln a}$$

This is useful because most calculators can only calculate logarithms to the base e (key `ln`) and 10 (key `log`). All other logarithms have to be converted to these bases.

7.2.6 Tasks

1. Solve for x:

   a) $1 = e^x$
   b) $8 = 2^x$
   c) $3 = 5e^x$
   d) $e = \frac{e^x}{e}$
   e) $9 = e^{cx}$
   f) $3 = \log_2 x$
   g) $0 = \log_{42} x$
   h) $0 = 5 \log_5 x$
   i) $9 = 3 \ln e^x$

2. Simplify:

   a) $\lg 2 + \lg 5$
   b) $\lg 5 + \lg 6 - \lg 3$
   c) $3 \ln a + 5 \ln b - \ln c$
   d) $2 \ln v - \ln v$
   e) $\frac{1}{2} \log_7 9 - \frac{1}{4} \log_7 81$
   f) $\log_3(x - 4) + \log_3(x + 4) = 3$
   g) $2 \log_2(4 - x) + 4 = \log_2(x + 5) - 1$
   h) $\log_5 x = \log_5 6 - 2 \log_5 3$

3. The growth of bacterial cultures can be described using the e function. The number of bacteria at time t is a function $N(t)$ that depends on the initial number of bacteria (i.e. the value $N_0 := N(0)$) and the growth rate k of the bacterium (constant). This results in the formula: $N(t) = N_0 e^{kt}$.

   For $N_0 = 100, k = 0.2$:

   a) How many bacteria are there at time $t = 5$ $(10, 100)$?
   b) At what time t will there be 500 (10000) bacteria?

7.3 Curve sketching, differentiation, integration

Author: Andreas Zöllner ‘Revision: Gerhard Gossen

A curve sketching helps us to “understand” a function. We get information about the shape of the graph (e.g. number and location of extrema and turning points) and about important points (e.g. zeros) of the function.
A curve discussion has a fixed sequence of steps. Some of these can be carried out by mathematical programmes, but for more complex functions, your own brainpower is required. The steps are: definition range, value range, zeros, (global) extrema, turning points, behaviour in infinity, poles and asymptotes. With the help of this data, the graph of the function can be sketched.

7.3.1 Domain

First, you should be clear about the domain $D(f)$ of the function f : for which values $x \in \mathbb{R}$ is $f(x)$ defined? For example, the function $\frac{1}{x-1}$ is not defined for $x = 1$ (division by 0!), and the logarithm function is only defined for positive values.
Isolated points at which f is not defined are called **gaps in the definition**. There are various types of gaps in the definition, but we will not go into them here.
The domain is given as a set. Examples of various domains are given in Table 7.1.

Definition range	**Description**
$D(f) = \mathbb{R}$	The definition range is the entire definition set.
$D(f) = \mathbb{R} \setminus \{c\}$	The function has a definition gap at the point c.
$D(f) = \{x \in \mathbb{R} \mid a < x < b \land x \geq c\}$	f is only defined in the intervals (a, b) and $[c, \infty)$

Table 7.1: Examples of domains

7.3.2 Range of values

The range of values $W(f)$ is all the values that the function f takes on, i.e. all values of $f(x)$ with $x \in D(f)$. The range of values is given as a set, analogous to the domain.
The range of a function can usually be determined by considering continuity, extrema, monotony and asymptotes.

7.3.3 Zeros

A point $x_0 \in D(f)$ is called a **zero** of the function f if

$$f(x_0) = 0,$$

holds.
To determine the zeros, we therefore have to find all solutions of the equation

$$f(x) \;=\; 0$$

.

7.3.4 Insert: Deriving a function

Since the derivative of the function is needed to determine the extrema and turning points, the following section briefly explains what the derivative is and how it is calculated. The first derivative f' indicates the slope of the tangents to the graph of f. To the left of a maximum, the slope is positive, to the right of it it is negative (see Fig. **??**). If the derivative has the value 0, this corresponds to a tangent with a slope of 0, i.e. a horizontal line. At a minimum, the slope of the tangents changes accordingly from negative to positive.

Derivative rules

1. Factor rule: $c \cdot f(x)(c \in \mathbb{R}, constant) \Rightarrow c \cdot f'(x)$
2. Sum rule: $f(x) + g(x) \Rightarrow f'(x) + g'(x)$
3. Product rule: $f(x)g(x) \Rightarrow f`(x)g(x) + f(x)g'(x)$
4. Quotient rule: $\dfrac{f(x)}{g(x)}(g(x) \neq 0) \Rightarrow \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
5. Chain rule: $f(g(x)) \Rightarrow f'(g(x)) \cdot g'(x)$

Important derivatives

Function	Derivative
c	0
x^n	$n \cdot x^{n-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\dfrac{1}{\cos^2 x}$
e^x	e^x
$\ln x$	$\frac{1}{x}$

7.3.5 Insert: Integrating a function

The inverse function of the derivative is integration. We are therefore looking for a function F whose derivative is the function f, i.e. $F' = f$. This function F is then called the primitive of f and we also write $\int f = F$. If we are looking for this integral for the function f in general, we also speak of an *indefinite integral*. However, if a specific interval of integration is sought in the domain of definition, this is called a *definite integral*. This definite integral then corresponds to the area between the graph of the function and the x-axis in the range of this interval. The primitive function is also required to calculate the definite integral and is then used in it:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Integration rules

1. Factor rule: $\int (c \cdot x)\, dx$ $(c \in \mathbb{R}, constant) \Rightarrow c \cdot \int f(x)\, dx$
2. Power rule: $\int x^n\, dx \Rightarrow \frac{1}{n+1} x^{n+1} + c$
3. Sum rule: $\int (f(x) + g(x))\, dx \Rightarrow \int f(x)\, dx + \int g(x)\, dx$

Important integrals

Function	primitive function
1	$x + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
e^x	$e^x + c$
$\ln x$	$x \cdot (\ln x - 1) + c$

7.3.6 Extrema

To determine the extrema of a function f, the first two derivatives of f must exist. A point x_0 is an extremum of f if

1. $f‘(x_0) = 0$ (*necessary condition*)
2. $f''(x_0) \neq 0$ (*sufficient condition*)

If $f''(x) > 0$ then there is a local minimum, if $f''(x) < 0$ then there is a local maximum.

The **global extrema** are obtained by additionally considering the behaviour of the function at the boundaries of the domain. For example, if $D(f) = \mathbb{R}$ then these are the values of $\lim_{x \to \pm\infty} f(x)$.

7.3.7 Turning points

At a *turning point* the curvature of the function graph changes, i.e. the graph changes from a left-hand curve to a right-hand curve or vice versa.
The necessary criterion for a turning point at x_0 is that the value of the second derivative becomes zero: $f''(x_0) = 0$. In addition, one of the following two conditions must be met:

The value of the third derivative is not equal to zero: $f'''(x_0) \neq 0$. However, the third derivative must exist.

The sign of the second derivative changes at x_0: If there is no third derivative or it is too difficult to calculate, the sign on both sides of x_0 must be compared.

The second derivative indicates the change in slope. If the second derivative is positive, the slope becomes steeper, so the graph curves to the left. A right-hand curve is formed when the slope of the tangent decreases, i.e. when the second derivative is negative.
A special form of a turning point is the *saddle point*, where both the first and second derivatives are zero. An example of this is shown in Figure 7.6. Here you can see that it is not enough to find a zero of the first derivative to determine the extrema; the second derivative must also be checked.

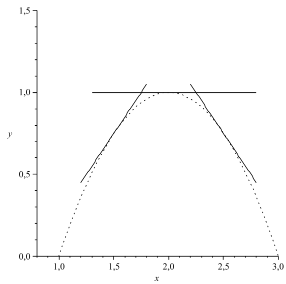


Figure 7.5: Tangents of the function $-(x-2)^2+1$ at the points $\frac{3}{2}, 2, \frac{5}{2}$

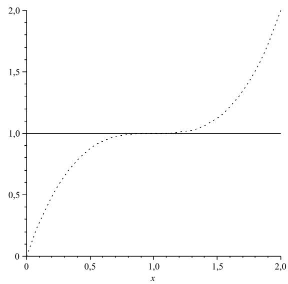


Figure 7.6: Saddle point of the function $(x-1)^3+1$ at the point $x = 1$

7.3.8 Behaviour at infinity, poles, asymptotes

The behaviour of the function f at infinity is understood to mean the limiting values

$$\lim_{x\to\infty} f(x) \qquad \text{bzw.} \qquad \lim_{x\to-\infty} f(x)\,,$$

provided that they exist.

The asymptotes describe the behaviour of the function f at infinity and at poles (a kind of gap in the definition) in more detail. The term "asymptotic" means "“approaching ”. An **asymptote** of the function f is a linear function

$$y = mx + n \quad \text{for certain } m, n \in \mathbb{R},$$

to which the function f approaches.

7.3.9 Graph of the function

With the help of this information, the graph of the function can now be drawn easily. To do this, you draw the zeros, extrema, inflection points and any asymptotes, which gives you the rough structure. If necessary, you can still calculate the function values for individual points to see, for example, the strength of the curvature.

7.3.10 Tasks

1. Calculate the derivatives of the following functions:

   a) $6x^2 - 5x + 7$
   b) $25x^4$
   c) $x(x - 7)$
   d) $(x^2 + 1)(x + 5)$
   e) $(x - 2)e^x$
   f) $\ln(x^3 - 9)$
   g) $(2x - 3)^5$
   h) $\sin(2x)$
   i) $\frac{1}{x^2 - 9}$

2. Calculate the following indefinite integrals:

   a) $\int (3x^2)dx$
   b) $\int (5x^3 - 2)dx$
   c) $\int (x - 4)(x + 1)dx$
   d) $\int \frac{2}{x} dx$
   e) $\int (e^{2x})dx$

3. Calculate the following definite integrals (area under the graph):

   a) $$\int_0^2 (3x^2)dx$$

   b) $$\int_1^5 (x + 1)dx$$

4. Perform a full curve sketching for these functions:

   a) $f(x) = -x^3 + 3x - 2$
   b) $g(x) = \frac{3x^2 - 12x}{4x^2 - 2}$

8 Vectors

Author: Gerhard Gossen R̈evision: Marko Rak, Melanie Pflaume

8.1 Definition

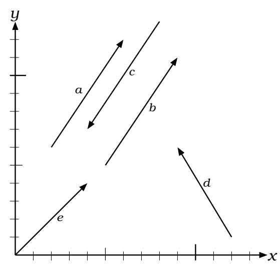
The vector

$$\overrightarrow{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

is an n-dimensional vector. The *components* $x_1, x_2, \ldots, x_n$ are real numbers. In the lecture, instead of $\overrightarrow{x}$, x is usually written.
Vectors of dimension 2 are geometrically interesting:

$$\overrightarrow{x} = \begin{pmatrix} x \\ y \end{pmatrix} \text{ and of dimension 3: } \overrightarrow{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

They can be represented as arrows pointing in a particular direction. They can also be interpreted as *displacements*. The zero vector $\overrightarrow{0}$ or 0 is the vector for which all components $x_1, x_2 \ldots x_n$ 0 are. The *position vector* of a point P is the vector between the origin of the coordinate system and P. A *scalar* is a single number of the same type as $x_1, x_2, \ldots, x_n$.



$$a = \begin{pmatrix} 4 \\ 6 \end{pmatrix} = b;\ c = \begin{pmatrix} -4 \\ -6 \end{pmatrix} = -1 \cdot a;\ d = \begin{pmatrix} -3 \\ 5 \end{pmatrix};\ e = \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

8.2 Operations

8.2.1 Addition and subtraction

Two vectors are added by adding the individual components:

$$x + y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

Subtraction is analogous:

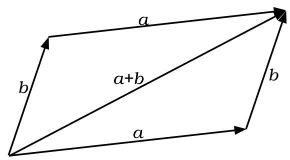
$$x - y = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_n - y_n \end{pmatrix}$$

Examples:

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ 11 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 7 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 6 \\ 7 \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix} \quad \begin{pmatrix} 3 \\ 7 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} \quad \begin{pmatrix} 12 \\ -5 \\ 0 \end{pmatrix} - \begin{pmatrix} -7 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 19 \\ -9 \\ -3 \end{pmatrix}$$

Geometrically, the addition of the vectors a and b corresponds to the displacement that results from moving first in the direction of a and then in the direction of b. As can be seen in the diagram, the addition is *commutative*, i.e. $a + b = b + a$.



8.2.2 Multiplication by a scalar

A vector is multiplied by a scalar by multiplying each individual component is multiplied by the scalar:

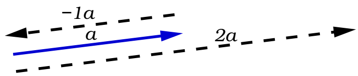
$$\lambda \cdot x = \lambda \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda \cdot x_1 \\ \lambda \cdot x_2 \\ \vdots \\ \lambda \cdot x_n \end{pmatrix}$$

Multiplication by the scalar 0 always results in the zero vector.

Examples:

$$1 \cdot \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix} \quad 2 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \quad -1 \cdot \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ -3 \end{pmatrix} \quad 0 \cdot \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Geometrically, multiplication corresponds to an *extension* by the factor λ.



Vector a, scaled with $\lambda = -1$ (top) and $\lambda = 2$ (bottom).

8.3 Linear combination

Any vector b that can be represented as the sum $b = \lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n$ is called a *linear combination* of the vectors $a_1, a_2, \ldots, a_n$. The λ_i are real numbers.

8.4 Linear dependence

The vectors $a_1, \ldots a_n$ are *linearly independent* if the equation

$$\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n = \overrightarrow{0}$$

has only the trivial solution $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. Otherwise the vectors are *linearly dependent.*

If two or more vectors are linearly dependent, then one vector can be represented as a linear combination of the other vectors.

Example: The vectors

$$\begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$$

are linearly dependent, since

$$1 \cdot \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 6 \\ 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} \text{ or } 1 \cdot \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} + \frac{1}{2} \cdot \begin{pmatrix} 6 \\ 2 \\ 8 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix} = \overrightarrow{0}$$

8.5 Magnitude of a vector

The magnitude $|a|$ of a vector is equal to the length of that vector. It is calculated as

$$|a| = \left| \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \right| = \sqrt{\sum_{i=1}^{n} a_i^2}$$

Examples:

$$\left| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| = \sqrt{1^2 + 0^2} = \sqrt{1} = 1$$

$$\left| \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$\left| \begin{pmatrix} 2 \\ -3 \\ 1 \\ 7 \\ -1 \end{pmatrix} \right| = \sqrt{2^2 + (-3)^2 + 1^2 + 7^2 + 1^2} = \sqrt{4 + 9 + 1 + 49 + 1} = \sqrt{64} = 8$$

8.6 Scalar product

The scalar product (a, b) of the two vectors a and b is the real number

$$(a, b) = |a||b| \cos \alpha$$

where α is the angle between the vectors (alternatively: $a \cdot b$).

The scalar product of two vectors of the nth order can also be calculated as follows:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i \cdot y_i$$

Usually, you want to check whether two vectors are orthogonal to each other. With $\cos(90°) = \cos\left(\frac{\pi}{2}\right) = 0$, we get:

$$\frac{(a, b)}{|a||b|} = 0$$

Examples:

1. Angle between $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$:

$$\cos\alpha = \frac{\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)}{\left|\begin{pmatrix}1\\0\end{pmatrix}\right|\left|\begin{pmatrix}0\\1\end{pmatrix}\right|} = 10 + 01/11 = 0\alpha = arccos(0) =$$

2. Angle between $a = \begin{pmatrix}-4\\2\\-2\end{pmatrix}$ and $b = \begin{pmatrix}10\\-5\\5\end{pmatrix}$

$$\begin{aligned}\cos\alpha &= \frac{(a,b)}{|a||b|}\\ &= \frac{-40+(-10)+(-10)}{\sqrt{24}\sqrt{150}} = \frac{-60}{\sqrt{4\cdot 6}\sqrt{25\cdot 6}}\\ &= -\frac{60}{2\sqrt{6}\cdot 5\sqrt{6}} = -\frac{60}{60} = -1\\ \alpha &= \arccos(-1) = \pi = 180^\circ\end{aligned}$$

8.7 Cross product

The *cross product* of two three-dimensional vectors a and b (both not equal to the zero vector) is a new vector. This is orthogonal to a and b. Notation: $a \times b$.
The cross product for 3-dimensional vectors is calculated as follows:

$$a \times b = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \times \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3 - a_3b_2\\a_3b_1 - a_1b_3\\a_1b_2 - a_2b_1\end{pmatrix}$$

Reminder: The value at the position $\bullet$ is given by $(1)\cdot(2) - (3)\cdot(4$, where $(1), \ldots, (4)$ can be taken from the formula.

$$\begin{pmatrix}\bullet\\ \\ \end{pmatrix} = \begin{pmatrix} \\1\\3\end{pmatrix} \times \begin{pmatrix} \\4\\2\end{pmatrix} \quad \begin{pmatrix} \\ \bullet\\ \end{pmatrix} = \begin{pmatrix}3\\ \\1\end{pmatrix} \times \begin{pmatrix}2\\ \\4\end{pmatrix} \quad \begin{pmatrix} \\ \\ \bullet\end{pmatrix} = \begin{pmatrix}1\\3\\ \end{pmatrix} \times \begin{pmatrix}4\\2\\ \end{pmatrix}$$

Examples:

$$\begin{pmatrix}1\\2\\3\end{pmatrix} \times \begin{pmatrix}4\\5\\6\end{pmatrix} = \begin{pmatrix}2\cdot 6 - 3\cdot 5\\3\cdot 4 - 1\cdot 6\\1\cdot 5 - 2\cdot 4\end{pmatrix} = \begin{pmatrix}-3\\6\\-3\end{pmatrix}$$

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\cdot 0 - 0\cdot 1\\0\cdot 0 - 1\cdot 0\\1\cdot 1 - 0\cdot 0\end{pmatrix} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

8.8 Tasks

1. Given:

$$a = \begin{pmatrix}2\\3\\-1\end{pmatrix}, b = \begin{pmatrix}-4\\1\\5\end{pmatrix}, c = \begin{pmatrix}-2\\-2\\-2\end{pmatrix}, d = \begin{pmatrix}7\\9\\1\end{pmatrix}$$

Calculate:

a) $a + b - c + d$

b) $d - c - b - a$

c) $3a - 2b + c$

d) $a - \frac{1}{2}c + (-3)b + 2d$

e) $2a - b + 5c - d$

f) $3a - 5b + 4c + 2d$

2. Calculate the length of the vectors:

a) $\begin{pmatrix}0\\1\\0\end{pmatrix}$

b) $\begin{pmatrix}2\\3\\2\end{pmatrix}$

c) $\begin{pmatrix}4\\3\\5\end{pmatrix}$

d) $\begin{pmatrix}-2\\2\\1\end{pmatrix}$

e) $\begin{pmatrix}3\\-3\end{pmatrix}$

f) $\begin{pmatrix}2\\-2\\2\\2\end{pmatrix}$

3. Determine the scalar product:

a) $\begin{pmatrix}0\\0\\1\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}$ b) $\begin{pmatrix}2\\-1\\-3\end{pmatrix}\cdot\begin{pmatrix}-2\\1\\3\end{pmatrix}$ c) $\begin{pmatrix}3\\4\\2\end{pmatrix}\cdot\begin{pmatrix}2\\-7\\5\end{pmatrix}$

4. Determine the included angle:

a) $\begin{pmatrix}1\\1\\0\end{pmatrix},\begin{pmatrix}0\\\sqrt{2}\\0\end{pmatrix}$ b) $\begin{pmatrix}3\\2\\1\end{pmatrix},\begin{pmatrix}-5\\1\\13\end{pmatrix}$

Calculate the cross product:

a) $\begin{pmatrix}2\\5\\3\end{pmatrix}\times\begin{pmatrix}-1\\2\\2\end{pmatrix}$ b) $\begin{pmatrix}1\\0\\2\end{pmatrix}\times\begin{pmatrix}3\\4\\7\end{pmatrix}$ c) $\begin{pmatrix}-2\\-3\\-1\end{pmatrix}\times\begin{pmatrix}-4\\-2\\-7\end{pmatrix}$

5. Check whether the vectors are linearly dependent. If so, express one of the vectors as a linear combination of the others. (Hint: Use the Gauss algorithm)

a) $\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}$

b) $\begin{pmatrix}3\\5\end{pmatrix},\begin{pmatrix}5\\3\end{pmatrix}$

c) $\begin{pmatrix}1\\2\end{pmatrix},\begin{pmatrix}7\\3\end{pmatrix},\begin{pmatrix}17\\5\end{pmatrix}$

d) $\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}2\\5\\0\end{pmatrix},\begin{pmatrix}3\\7\\1\end{pmatrix}$

e) $\begin{pmatrix}7\\2\\5\end{pmatrix},\begin{pmatrix}3\\-5\\8\end{pmatrix},\begin{pmatrix}10\\-3\\13\end{pmatrix}$

f) $\begin{pmatrix}-2\\-3\\4\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix},\begin{pmatrix}7\\6\\5\end{pmatrix}$

g) $\begin{pmatrix}3\\7\\5\end{pmatrix},\begin{pmatrix}-2\\5\\1\end{pmatrix},\begin{pmatrix}-7\\3\\-3\end{pmatrix}$

h) $\begin{pmatrix}1\\0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\3\\2\end{pmatrix},\begin{pmatrix}1\\1\\0\\1\end{pmatrix}$

i) $\begin{pmatrix}1\\0\\2\\1\end{pmatrix},\begin{pmatrix}1\\0\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\\1\end{pmatrix},\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$

9 Matrices

Author: Martin Glauer

A matrix is nothing more than a tabular arrangement of elements from a set K with a multiplication $\cdot$ and an addition $+$. They are used in many areas of computer science, for example to calculate rotations of objects. A matrix with m rows (horizontal) and n columns (vertical) is also called an $m \times n$ matrix (read: *'m cross n matrix'*) and, correspondingly, the space that contains all $m \times n$ matrices is denoted by $K^{m \times n}$. Similarly, the individual entries of a matrix A are given in the form $A_{row,column}$:

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{pmatrix} in K^{m \times n}$$

[1]

9.1 Scalar multiplication

A matrix $A \in K^{m \times n}$ can be multiplied (in the same way as the scalar product for vectors) by an element λ from the space from which its entries originate.
To do this, all entries of the matrix are multiplied by this element

$$\lambda \cdot \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} = \begin{pmatrix} \lambda \cdot A_{1,1} & \cdots & \lambda \cdot A_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda \cdot A_{m,1} & \cdots & \lambda \cdot A_{m,n} \end{pmatrix}$$

[1] This set must form a field. A structure that you will get to know in the course of your studies. Informally speaking, a field is a set such that the operations $+$ and $\cdot$ are similar to addition and multiplication in the rational numbers $\mathbb{Q}$ (and are therefore also reversibly!).

9.2 Addition

To add two matrices, the entries in the same positions are added:

$$\begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1} & \cdots & A_{m,n} \end{pmatrix} + \begin{pmatrix} B_{1,1} & \cdots & B_{1,n} \\ \vdots & \ddots & \vdots \\ B_{m,1} & \cdots & B_{m,n} \end{pmatrix} = \begin{pmatrix} A_{1,1}+B_{1,1} & \cdots & A_{1,n}+B_{1,n} \\ \vdots & \ddots & \vdots \\ A_{m,1}+B_{m,1} & \cdots & A_{m,n}+B_{m,n} \end{pmatrix}$$

9.2.1 Calculation rules

For all matrices $A, B \in K^{m\times n}$and scalars $\lambda \in K$ the following applies:

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $\lambda(A + B) = \lambda A + \lambda B$

9.3 Transposition

The transpose A^T of a matrix $A \in K^{m\times n}$ is the mirror image about the main diagonal $A_{1,1}$, $A_{2,2}$, ..., $A_{n,n}$: for example, if $m > n$ (i.e. if A has more rows than columns):

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \\ \vdots & \vdots & & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{pmatrix}^T = \begin{pmatrix} A_{1,1} & A_{2,1} & \cdots & A_{n,1} & \cdots & A_{m,1} \\ A_{1,2} & A_{2,2} & \cdots & A_{n,2} & \cdots & A_{m,2} \\ \vdots & \vdots & & \vdots & & \vdots \\ A_{1,n} & A_{2,n} & \cdots & A_{n,n} & \cdots & A_{m,n} \end{pmatrix}$$

9.3.1 Calculation rules

The following applies to all matrices A, B and scalars $\lambda \in K$:

- $(A + B)^T = A^T + B^T$
- $(\lambda A)^T = \lambda A^T$
- $(A^T)^T = A$

9.4 Multiplication

If you want to calculate the product $C \in K^{k \times n}$ for two matrices $A \in K^{k \times m}, B \in K^{m \times n}$, this is a little more complex:

$$C_{i,j} = \sum_{t=1}^{m} A_{i,t} \cdot B_{t,j}$$

Important! The number of columns in the left-hand matrix must match the number of rows in the right-hand matrix. Furthermore, multiplication is generally not reversible (commutative): $A \cdot B \neq B \cdot A$.

9.4.1 The Falk scheme:

The Falk scheme (also known as the Falk diagram) offers a simplified form of representation in which both matrices are written in the form shown in Figure **??**.

$$\begin{array}{cc} & \begin{pmatrix} B_{1,1} & \cdots & B_{1,n} \\ B_{2,1} & \cdots & B_{2,n} \\ \vdots & \ddots & \vdots \\ B_{m-1,1} & \cdots & B_{m-1,n} \\ B_{m,1} & \cdots & B_{m,n} \end{pmatrix} \\ \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,m-1} & A_{1,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{k,1} & A_{k,2} & \cdots & A_{k,m-1} & A_{k,m} \end{pmatrix} & \begin{pmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{k,1} & \cdots & C_{k,n} \end{pmatrix} \end{array}$$

Figure 9.1: The Falk scheme

To calculate $C_{i,j}$, only the ith row of A and the jth column of B are considered. First, the innermost entries are multiplied $(A_{i,m} \cdot B_{m,j})$, which is iteratively repeated all entries from the inside out: $A_{i,m-1} \cdot B_{m-1,j}$, $A_{i,m-2} \cdot B_{m-2,j}$, $\cdots$, $A_{i,1} \cdot B_{1,j}$. The entry $C_{i,j}$ is then obtained from the sum of these products:

$$C_{i,j} = A_{i,m} \cdot B_{m,j} + A_{i,m-1} \cdot B_{m-1,j} + \cdots + A_{i,1} \cdot B_{1,j}$$

9.4.2 Calculation rules

- $(A \cdot B) \cdot C = A \cdot (B \cdot C)$
- $(A + B) \cdot C = A \cdot C + B \cdot C$
- $A \cdot (B + C) = A \cdot B + A \cdot C$
- $(A \cdot B)^T = B^T \cdot A^T$

9.4.3 Matrices as systems of equations

A system of equations of the form

$$\begin{array}{ccccccccc} A_{1,1}x_1 & + & A_{1,2}x_2 & + & \cdots & + & A_{1,n}x_n & = & b_1 \\ A_{2,1}x_1 & + & A_{2,2}x_2 & + & \cdots & + & A_{2,n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ A_{m,1}x_1 & + & A_{m,2}x_2 & + & \cdots & + & A_{m,n}x_n & = & b_m \end{array}$$

can also be represented using vectors and matrices:

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

i.e.:

$A \cdot \overrightarrow{x} = \overrightarrow{b}$

9.5 Inverse Matrix

A square matrix $A \in K^{n \times n}$is **invertible** if there is a matrix $A^{-1} \in K^{n \times n}$ such that:

$$A \cdot A^{-1} = A^{-1} \cdot A = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

In this case, the matrix $A^{-1}$is called the **inverse of A matrix**. A matrix for which there is no inverse is called not invertible. The inverse matrix is determined in a similar way to the Gaussian method:

$$
: A_{1,1} \quad \left(\begin{array}{ccc|cccc}
A_{1,1} & \cdots & A_{1,n} & 1 & 0 & \cdots & 0 \\
A_{2,1} & \cdots & A_{2,n} & 0 & 1 & & 0 \\
\vdots & & \vdots & \vdots & & \ddots & \vdots \\
A_{n,1} & \cdots & A_{n,n} & 0 & \cdots & 0 & 1
\end{array}\right)
$$

$$\downarrow$$

$$
\cdot(-A_{2,1}); + \quad \left(\begin{array}{ccc|cccc}
1 & \cdots & \frac{A_{1,n}}{A_{1,1}} & \frac{1}{A_{1,1}} & 0 & \cdots & 0 \\
A_{2,1} & \cdots & A_{2,n} & 0 & 1 & & 0 \\
\vdots & & \vdots & \vdots & & \ddots & \vdots \\
A_{n,1} & \cdots & A_{n,n} & 0 & \cdots & 0 & 1
\end{array}\right)
$$

$$\downarrow$$

$$
\left(\begin{array}{ccc|cccc}
1 & \cdots & \frac{A_{1,n}}{A_{1,1}} & \frac{1}{A_{1,1}} & 0 & \cdots & 0 \\
0 & \cdots & A_{2,n} - \frac{A_{2,1} \cdot A_{1,n}}{A_{1,1}} & 0 - \frac{A_{2,1}}{A_{1,1}} & 1 & & 0 \\
\vdots & & \vdots & \vdots & & \ddots & \vdots \\
A_{n,1} & \cdots & A_{n,n} & 0 & \cdots & 0 & 1
\end{array}\right)
$$

$$\downarrow$$

$$\vdots$$

$$\downarrow$$

$$
\left(\begin{array}{cccc|cccc}
1 & 0 & \cdots & 0 & B_{1,1} & \cdots & \cdots & B_{1,n} \\
0 & 1 & & 0 & \vdots & \ddots & & \vdots \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & \cdots & 0 & 1 & B_{n,1} & \cdots & \cdots & B_{n,n}
\end{array}\right)
$$

The matrix on the right is then the inverse matrix ($B = A^{-1}$). **Important:** If there is a 0 at the beginning of the first step, this row is swapped with a row further down on both sides.

9.5.1 Example

Let the matrix $A = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 1 & 3 \\ 2 & -1 & 2 \end{pmatrix}$.

Divide the first row by 2

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 4 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 & 0 & 1 \end{array}\right)$$

Subtract the first row from the second and subtract twice the first row from the third row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right)$$

Add the second row to the third row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & -1 & -2 & -1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -1 & -\frac{3}{2} & 1 & 1 \end{array}\right)$$

Multiply the third row by -1

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & -1 & -\frac{3}{2} & 1 & 1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -1 & -1 \end{array}\right)$$

Now we have the so-called upper triangular form. Now the procedure must be repeated from the bottom up for the elements above the diagonal. Subtract the third row from the second row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 1 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} & -1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & -1 & -1 \end{array}\right)$$

Subtract twice the third row from the first row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & -1 & -1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{5}{2} & 2 & 2 \\ 0 & 1 & 0 & -2 & 2 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & -1 & -1 \end{array}\right)$$

This is the inverse:

$$A^{-1} = \begin{pmatrix} -\frac{5}{2} & 2 & 2 \\ -2 & 2 & 1 \\ \frac{3}{2} & -1 & -1 \end{pmatrix}$$

9.5.2 Calculation rules

For **invertible** matrices A, B and $\lambda \in K \backslash \{0\}$:

- $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$
- $\left(A^{-1}\right)^{T} = \left(A^{T}\right)^{-1}$

- $\left(A^{-1}\right)^{-1} = A$
- $(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$

9.6 Tasks

9.6.1 Task 1

The following matrices are given:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Calculate:

1. A^T, B^T and C^T
2. $B + C$
3. $A \cdot B$
4. $B \cdot A^T$
5. $B^T \cdot A^T$
6. * B^{-1} (if possible)
7. * C^{-1} (if possible)

9.6.2 Task 2

If possible, invert the following matrices:

1. $\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$
2. $\begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 4 \\ 3 & 0 & 2 \end{pmatrix}$
3. $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 5 & 0 \\ 3 & 9 & 6 \end{pmatrix}$

10 Complex Numbers

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10.1 History

There is no real number $x \in \mathbb{R}$ that satisfies the equation

$$x^2 = -1$$

. To formulate solutions to this equation, a number range extension must be carried out. This is why R. Bombielli introduced the symbol $\sqrt{-1}$ in the middle of the 16th century, for which L. Euler later wrote i. This **imaginary unit** is defined as a solution to the equation

$$\mathrm{i}^2 = -1\,.$$

10.2 Cartesian representation

A **complex number** z is a symbol of the form

$$z = x + \mathrm{i}y \qquad \text{with } x, y \in \mathbb{R}.$$

The **set of complex numbers** is denoted by $\mathbb{C}$;

$$\mathbb{C} = \{\, x + \mathrm{i}y \mid x, y \in \mathbb{R} \,\}\,.$$

From this **Cartesian representation** $z = x + \mathrm{i}y$ of the complex number $z \in \mathbb{C}$ we can visualise z as an ordered pair or a two-dimensional vector $(x, y) \in \mathbb{R}^2$, i.e. a point in the **Gaussian number plane**.
For a complex number $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$

$$\mathrm{Re}(z) := x \qquad \text{and} \qquad \mathrm{Im}(z) := y$$

denote the **real part** and **imaginary part** of z. Thus

$$z = \mathrm{Re}(z) + \mathrm{i} \cdot \mathrm{Im}(z)\,.$$

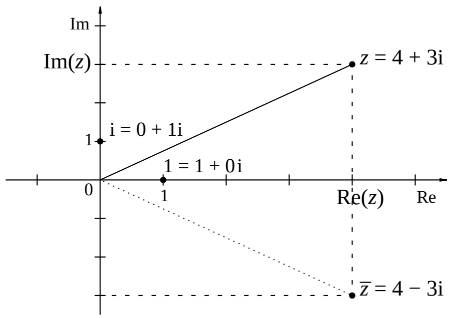
The set of real numbers, $\mathbb{R}$, is obviously a subset of the complex numbers $z \in \mathbb{C}$ with $\mathrm{Im}(z) = 0$. The complex numbers with $\mathrm{Re}(z) = 0$ are called the *purely imaginary numbers*.

10.3 Calculation operations

Complex numbers are calculated according to the usual rules of arithmetic in the real numbers. In doing so, i is treated like a variable for which $\mathrm{i}^2 = -1$ applies, i.e. powers i^k that occur during the calculation are reduced to $\mathrm{i} = \mathrm{i}^1$ again, so that a Cartesian representation of a complex number is again the result of the calculation. The following concept is also needed here: the **conjugate complex number** $\overline{z}$ of $z = x + \mathrm{i}y$ is the complex number

$$\overline{z} \;=\; \overline{x + \mathrm{i}y} \;:=\; x - \mathrm{i}y \qquad \text{for } x, y \in \mathbb{R}.$$

Graphically interpreted in the Gaussian number plane, this corresponds to reflection on the real axis.



The basic arithmetic operations can now be performed. Let $a, b, c, d \in \mathbb{R}$.

- Addition. This is performed component by component:

$$(a + \mathrm{i}b) \;+\; (c + \mathrm{i}d) \;=\; (a + c) + \mathrm{i}(b + d)$$

- Subtraction. This is also done component by component:

$$(a + \mathrm{i}b) \;-\; (c + \mathrm{i}d) \;=\; (a - c) + \mathrm{i}(b - d)$$

- Multiplication. This is expanded:

$$(a + \mathrm{i}b) \cdot (c + \mathrm{i}d) \;=\; ac + \mathrm{i}bc + \mathrm{i}ad + \mathrm{i}^2 bd \;=\; (ac - bd) + \mathrm{i}(bc + ad)$$

- Division. The denominator is made real by extending the fraction with the conjugate complex number of the denominator:

$$\begin{aligned} \frac{a + \mathrm{i}b}{c + \mathrm{i}d} &= \frac{(a + \mathrm{i}b)(c - \mathrm{i}d)}{(c + \mathrm{i}d)(c - \mathrm{i}d)} = \frac{(ac + bd) + \mathrm{i}(bc - ad)}{c^2 + d^2} \\ &= \frac{ac + bd}{c^2 + d^2} + \mathrm{i}\,\frac{bc - ad}{c^2 + d^2} \end{aligned}$$

- Exponentiation with exponent $n \in \mathbb{N}$. Repeated multiplication is performed. The following applies

$$(a + \mathrm{i}b)^0 \;:=\; 1 \quad \text{and} \quad (a + \mathrm{i}b)^{n+1} \;=\; (a + \mathrm{i}b) \cdot (a + \mathrm{i}b)^n.$$

The integral powers of i are important here. For $n \in \mathbb{Z}$,

$$\mathrm{i}^n \;=\; \begin{cases} 1\,, & \text{if } n \text{ leaves a remainder of 0 when divided by 4} \\ \mathrm{i}\,, & \text{if } n \text{ leaves a remainder of 1 when divided by 4} \\ -1\,, & \text{if } n \text{ leaves the remainder 2 when divided by 4} \\ -\mathrm{i}\,, & \text{if } n \text{ leaves the remainder 3 when divided by 4} \end{cases}$$

10.4 Euler's representation

For a complex number $z = x + \mathrm{i}y$ with $x, y \in \mathbb{R}$ we consider its representation as a vector in the Gaussian number plane in *polar coordinates*,

$$z \;=\; r\,(\cos\varphi + \mathrm{i}\sin\varphi) \qquad \text{with } r \geq 0 \text{ and } -\pi < \varphi \leq \pi.$$

The **magnitude** $|z|$ of z is

$$|z| \;:=\; \sqrt{x^2 + y^2} \;=\; \sqrt{z \cdot \overline{z}} \;=\; r \;\geq\; 0$$

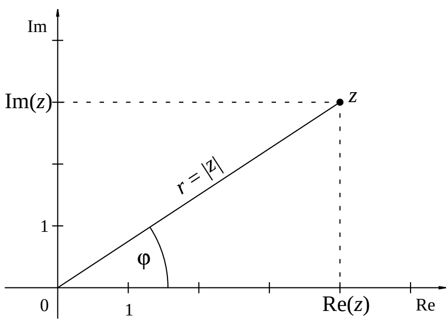
and the **(principal) argument** $\arg(z)$ of z is

$$\arg(z) \;=\; \varphi \;\; \text{with} \;\; -\pi < \varphi \leq \pi.$$

The angles $\varphi + 2k\pi$ for $k \in \mathbb{Z}$ are called the **arguments** of z. Note that all of these angles determine one and the same complex number z, i.e.,

$$z = r(\cos(\varphi + 2k\pi) + i\sin(\varphi + 2k\pi)) \text{ for all } k \in \mathbb{Z},$$

and that the principal argument is uniquely determined by the requirement that φ is in the interval $-\pi \leq \varphi \leq \pi$.



Euler's formula gives the **Euler form** of a complex number $z = x + \mathrm{i}y$ for $x, y \in \mathbb{R}$,

$$z = r\mathrm{e}^{\mathrm{i}\varphi} \qquad \text{with } r = |z| \text{ and } \varphi = \arg z.$$

This representation simplifies some calculations with complex numbers. Let $r, s \geq 0$ and $\varphi, \psi \in (-\pi, \pi]$. Then, using the power laws, the following applies:

- Multiplication: $r\mathrm{e}^{\mathrm{i}\varphi} \cdot s\mathrm{e}^{\mathrm{i}\psi} = (rs)\,\mathrm{e}^{\mathrm{i}(\varphi+\psi)}$
- Division: For $s > 0$ we have $\dfrac{r\mathrm{e}^{\mathrm{i}\varphi}}{s\mathrm{e}^{\mathrm{i}\psi}} = \dfrac{r}{s}\,\mathrm{e}^{\mathrm{i}(\varphi-\psi)}$

Now the basic arithmetic operations in the Gaussian number plane can be interpreted geometrically.

- Addition and subtraction are the usual (i.e. component-wise) operations for (two-dimensional) vectors.
- In multiplication, the absolute values of the two operands are multiplied and the arguments added. This is therefore a so-called *rotation stretch*.
- The transition from z to the complex conjugate number $\overline{z}$ corresponds to a reflection about the real axis.

10.5 Conversion between Cartesian and polar coordinates

First, we should remember the connection between the angular measures. A full circle of $360°$ corresponds to 2π. Thus

$$\varphi \text{ in degrees} = (\varphi \text{ in radians}) \cdot \frac{180°}{\pi}.$$

In particular, $90° = \pi/2$ and $180° = \pi$.
Given a point $z = (x, y) \in \mathbb{R}^2$ (in the Cartesian coordinate system), its polar coordinates (r, φ) are given by

$$r = |z| = \sqrt{x^2 + y^2} \geq 0$$

and φ as the solution of the system of equations

$$r\cos\varphi = x \quad \text{and} \quad r\sin\varphi = y \qquad \text{with} \qquad \varphi \in (-\pi, \pi]\,.$$

This trigonometric system of equations can be solved for $x \neq 0$ by finding a solution ψ of $\tan\psi = y/x$, for example

$$\psi = \arctan\left(\frac{y}{x}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

and then, using the signs of x and y, ensuring that the angle points in the correct quadrant, i.e. $\varphi = \psi + k\pi$ with the correct value of $k \in \mathbb{Z}$.

x	y	$\varphi =$
≥ 0	$= 0$	0
< 0	$= 0$	π
$= 0$	> 0	$\pi/2$
$= 0$	< 0	$-\pi/2$

x	y	$\varphi \in$
> 0	> 0	$(0, \pi/2)$
> 0	< 0	$(-\pi/2, 0)$
< 0	> 0	$(\pi/2, \pi)$
< 0	< 0	$(-\pi, -\pi/2)$

Sign schemes:

$$\sin : \begin{array}{c|c} + & + \\ \hline - & - \end{array} \qquad \cos : \begin{array}{c|c} - & + \\ \hline - & + \end{array} \qquad \tan : \begin{array}{c|c} - & + \\ \hline + & - \end{array}$$

Given a point in polar coordinates $z = (r, \varphi) \in [0, \infty) \times (-\pi, \pi]$, its cartesian coordinates (x, y) are given by

$$x = r\cos\varphi \qquad \text{and} \qquad y = r\sin\varphi\,.$$

10.6 Tasks

Calculate

1. $(1 + 2\mathrm{i}) + (4 - 3\mathrm{i})$, $(2 + 4\mathrm{i}) + 3$, $(4 + 2\mathrm{i}) - 2\mathrm{i}$
2. $(1 + 2\mathrm{i}) \cdot (4 - 3\mathrm{i})$, $(3 + 2\mathrm{i}) \cdot (3 - 2\mathrm{i}$, $(1 + 3\mathrm{i}) \cdot (-1 + 3\mathrm{i})$
3. $\frac{1 + 2\mathrm{i}}{4 - 3\mathrm{i}}, \frac{3 + 2\mathrm{i}}{3 - 2\mathrm{i}}, \frac{1 + 3\mathrm{i}}{-1 + 3\mathrm{i}}$

Exercise The complex numbers are given by

$$z_1 = -2i \qquad z_2 = -3 \qquad z_3 = -1 + 2\mathrm{i} \quad z_4 = -4 - 3\mathrm{i}$$
$$z_5 = e^{\pi/4} z_6 = e^{\mathrm{i}\pi/4} \quad z_7 = 2\mathrm{e}^{-\frac{3\pi}{4}\mathrm{i}} z_8 = -\frac{1}{2}\mathrm{e}^{\mathrm{i}\cdot 3\pi/2}$$

1. Plot the numbers in the Gaussian number plane.
2. Calculate their absolute value and their principal argument.
3. Convert from the cartesian to the Euler form and vice versa.

B. G. Teubner Leipzig, 1996.